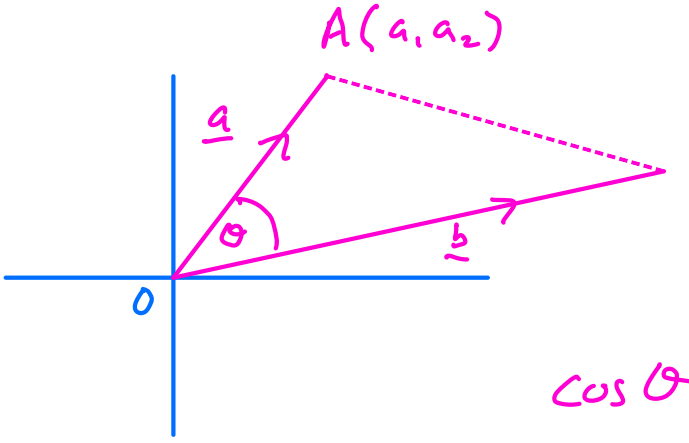
Angle Between Two Vectors



$$\overrightarrow{AB} = -\underline{a} + \underline{b} = \begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \end{pmatrix}$$

$$\cos\theta = \frac{|\overrightarrow{OA}|^2 + |\overrightarrow{OB}|^2 - |AB|^2}{2\,|\overrightarrow{OA}|\,|\overrightarrow{OB}|}$$

$$\cos\theta = \frac{\left(\sqrt{a_1^2 + a_2^2}\right)^2 + \left(\sqrt{b_1^2 + b_2^2}\right)^2 - \left(\sqrt{(b_1 - a_1)^2 + (b_2 - a_2)^2}\right)^2}{2\,|\overrightarrow{OA}|\,|\overrightarrow{OB}|}$$

$$\cos\theta = \frac{\cancel{a_1^2} + \cancel{a_2^2} + \cancel{b_1^2} + \cancel{b_2^2} - \left[\cancel{b_1^2} + \cancel{a_1^2} - 2a_1b_1 + \cancel{b_2^2} + \cancel{a_2^2} - 2a_2b_2\right]}{2\,|\overrightarrow{OA}|\,|\overrightarrow{OB}|}$$

$$\cos\theta = \frac{2a_1b_1 + 2a_2b_2}{2\,|\underline{a}|\,|\underline{b}|} = \frac{a_1b_1 + a_2b_2}{|\underline{a}|\,|\underline{b}|}$$

Aside

$\underline{a} \cdot \underline{b}$ is called the scalar product of $\underline{a}$ and $\underline{b}$ (or the dot product)

$\underline{a} \times \underline{b}$ is called the vector product of $\underline{a}$ and $\underline{b}$ (or the cross product)

$$\underline{a} \cdot \underline{b} = |\underline{a}|\,|\underline{b}|\cos\theta$$

$$\underline{a} \times \underline{b} = |\underline{a}|\,|\underline{b}|\sin\theta\,\underline{j}$$ where $\underline{j}$ is a unit

vector $\underline{1}$ to $\underline{a}$ and $\underline{b}$

In 2 dimensions $\underline{a} \cdot \underline{b} = a_1 b_1 + a_2 b_2$

In 3 dimensions $\underline{a} \cdot \underline{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$

Examples if $\underline{a} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ $\underline{b} = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$

$$\underline{a} \cdot \underline{b} = 2 \times -1 + 3 \times 4 = 10$$

if $\underline{a} = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}$ and $\underline{b} = \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix}$

$$\begin{aligned} \underline{a} \cdot \underline{b} &= 3 \times 2 + 4(-3) + 5 \times 6 \\ &= 6 - 12 + 30 \\ &= 24 \end{aligned}$$

$$\cos\theta = \frac{\underline{a} \cdot \underline{b}}{|\underline{a}||\underline{b}|}$$

Example If $\underline{a} = \begin{pmatrix} 4 \\ 7 \end{pmatrix}$ $\underline{b} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$

Find the angle between $\underline{a}$ and $\underline{b}$

$$\cos\theta = \frac{\begin{pmatrix} 4 \\ 7 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -1 \end{pmatrix}}{\left|\begin{pmatrix} 4 \\ 7 \end{pmatrix}\right| \left|\begin{pmatrix} 3 \\ -1 \end{pmatrix}\right|} = \frac{12 - 7}{\sqrt{4^2+7^2}\sqrt{3^2+(-1)^2}}$$

$$\cos\theta = \frac{5}{\sqrt{65}\sqrt{10}}$$

$$\theta = \cos^{-1}\frac{5}{\sqrt{650}} = 78.7^\circ$$

Modelling With Vectors

Exercise 11F

1 d) $\underline{v} = (-7\underline{i} + 4\underline{j})\ cm\ s^{-1}$

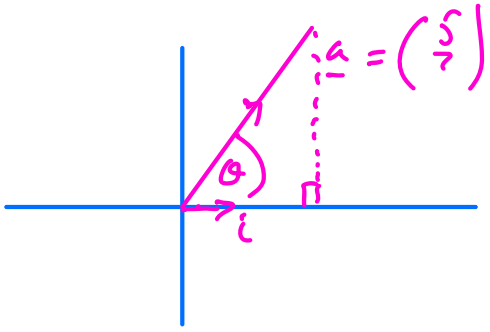
speed $= \sqrt{(-7)^2 + 4^2} = \sqrt{65}\ cm\ s^{-1}$

2 d) $\underline{v} = (-4\underline{i} - 7\underline{j})\ cm\ s^{-1}$ for 120s

distance $\sqrt{(-4)^2 + (-7)^2} \times 120 = 120\sqrt{65}\ cm$

5) N2L $\underline{F} = m\underline{a}$

$\underline{a} = (5\underline{i} + 7\underline{j})\ m\ s^{-2}$ or $\begin{pmatrix} 5 \\ 7 \end{pmatrix}\ m\ s^{-2}$



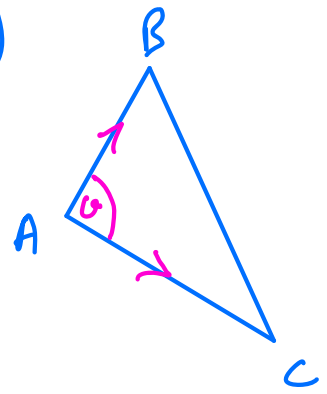
$\sin\theta = \frac{7}{\sqrt{5^2+7^2}}$ $\theta = 54.5^\circ$

b) $\underline{F} = m\underline{a}$

$\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = 0.3\begin{pmatrix} 5 \\ 7 \end{pmatrix} \Rightarrow \underline{F} = \begin{pmatrix} 1.5 \\ 2.1 \end{pmatrix}\ N$

$|\underline{F}| = \sqrt{1.5^2 + 2.1^2} = 2.58N$

7)



$\vec{AB} = \begin{pmatrix} 30 \\ 40 \end{pmatrix}$ $\vec{AC} = \begin{pmatrix} 40 \\ -60 \end{pmatrix}$

a) $\vec{BC} = \vec{BA} + \vec{AC}$

$= \begin{pmatrix} -30 \\ -40 \end{pmatrix} + \begin{pmatrix} 40 \\ -60 \end{pmatrix}$

$\vec{BC} = \begin{pmatrix} 10 \\ -100 \end{pmatrix}$

b) $\cos\theta = \dfrac{(\vec{AB}).(\vec{AC})}{|\vec{AB}||\vec{AC}|} = \dfrac{30\times40 + 40(-60)}{\sqrt{30^2+40^2}\ \sqrt{40^2+(-60)^2}}$

$= \dfrac{-1200}{50 \times 20\sqrt{13}}$

$\theta = \cos^{-1}\left(\dfrac{-1200}{1000\sqrt{13}}\right)$

$\theta = 109.4^\circ$

---

c) $\text{Area} = \frac{1}{2}ab\sin C$

$= \frac{1}{2}|\vec{AB}||\vec{AC}|\sin 109.44^\circ$

$= \frac{1}{2} \times 50 \times 20\sqrt{13} \times \sin 109.44^\circ$

$= 1700.000365$

$= 1700\ m^2$

Hwk Q1a Q2a, Q3a, Q4, Q6, Q8