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9. **Figure 3**

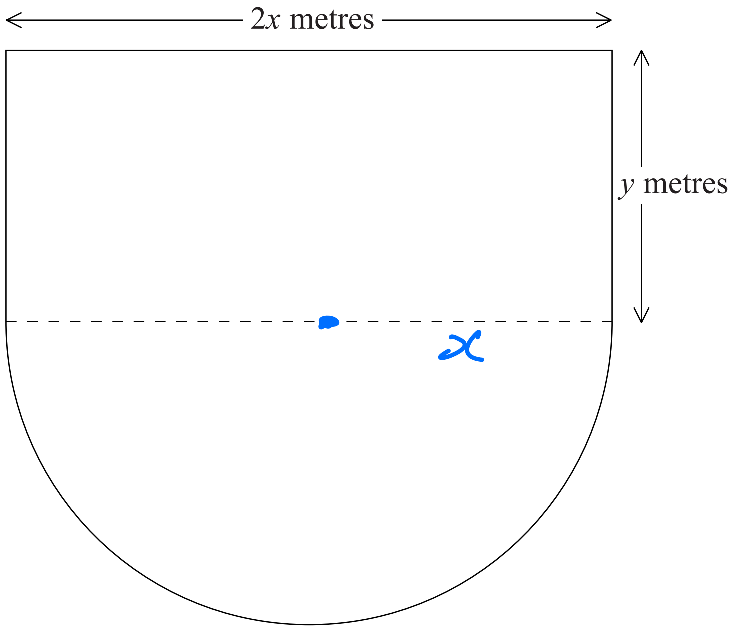

Figure 3 shows the plan of a stage in the shape of a rectangle joined to a semicircle. The length of the rectangular part is $2x$ metres and the width is y metres. The diameter of the semicircular part is $2x$ metres. The perimeter of the stage is 80 m.

(a) Show that the area, A m^2, of the stage is given by

$$A = 80x - \left(2 + \frac{\pi}{2}\right)x^2.$$

(4)

(b) Use calculus to find the value of x at which A has a stationary value.

(4)

(c) Prove that the value of x you found in part (b) gives the maximum value of A.

(2)

(d) Calculate, to the nearest m^2, the maximum area of the stage.

(2)

a) $P = 2x + y + y + \frac{2\pi x}{2}$

$P = 2x + 2y + \pi x$

$80 = (2+\pi)x + 2y$

$80 - (2+\pi)x = 2y$

$$y = 40 - \frac{(2+\pi)x}{2}$$

$$\text{Area} = 2xy + \frac{\pi x^2}{2}$$

$$\text{Area} = 2x\left[40 - \frac{(2+\pi)x}{2}\right] + \frac{\pi x^2}{2}$$

$$= 80x - (2+\pi)x^2 + \frac{\pi x^2}{2}$$

$$= 80x - 2x^2 - \pi x^2 + \frac{\pi x^2}{2}$$

$$= 80x - 2x^2 - \frac{\pi x^2}{2}$$

$$A = 80x - \left(2 + \frac{\pi}{2}\right)x^2$$

---

b)

$$\frac{dA}{dx} = 80 - 2\left(2 + \frac{\pi}{2}\right)x$$

At st.pt. $\frac{dA}{dx} = 0$

$$80 - 2\left(2 + \frac{\pi}{2}\right)x = 0$$

$$80 = 2\left(2 + \frac{\pi}{2}\right)x$$

$$40 = \left(2 + \frac{\pi}{2}\right)x$$

$$\frac{40}{\left(2 + \frac{\pi}{2}\right)} = x$$

$$x = 11.20198$$

$x = 11.2$ m to 3 s.f.

c) $\dfrac{d^2A}{dx^2} = -2\left(2+\frac{\pi}{2}\right) < 0$

$\therefore$ Maximum at $x = 11.2\,m$

---

d) $A = 80 \times 11.2 - \left(2+\frac{\pi}{2}\right) \times 11.2^2$

$A = 448.079$

$A = 448\,m^2$ to n'est m^2

Leave blank

8. A diesel lorry is driven from Birmingham to Bury at a steady speed of v kilometres per hour. The total cost of the journey, £C, is given by

$$C = \frac{1400}{v} + \frac{2v}{7}.$$

(a) Find the value of v for which C is a minimum.

(5)

(b) Find $\frac{d^2C}{dv^2}$ and hence verify that C is a minimum for this value of v.

(2)

(c) Calculate the minimum total cost of the journey.

(2)

a) $C = 1400v^{-1} + \frac{2v}{7}$

$\frac{dC}{dv} = -1400v^{-2} + \frac{2}{7}$

Min when $\frac{dC}{dv} = 0 \Rightarrow -\frac{1400}{v^2} + \frac{2}{7} = 0$

$\frac{1400}{v^2} = \frac{2}{7}$

$9800 = 2v^2$

$4900 = v^2$

$\sqrt{4900} = v$

$70 = v$

$v = 70$ km/h

b) $\frac{d^2C}{dv^2} = 2800v^{-3}$

N 2 4 3 2 2 A 0 1 6 2 4

$$\frac{d^2C}{dv^2} = \frac{2800}{v^3}$$

when $v = 70$, $\frac{d^2C}{dv^2} = \frac{2800}{70^3} > 0$ $\therefore$ minimum

---

c) $C = \frac{1400}{70} + \frac{2 \times 70}{7}$

$C = 20 + 20$

$C = £40$

Leave blank

10. The volume V cm^3 of a box, of height x cm, is given by

$$V = 4x(5-x)^2, \qquad 0 < x < 5$$

(a) Find $\frac{dV}{dx}$. **(4)**

(b) Hence find the maximum volume of the box. **(4)**

(c) Use calculus to justify that the volume that you found in part (b) is a maximum. **(2)**

a) $V = 4x(25 - 10x + x^2)$

$V = 100x - 40x^2 + 4x^3$

$\frac{dV}{dx} = 100 - 80x + 12x^2$

b) Max when $\frac{dV}{dx} = 0$

$\Rightarrow \quad 12x^2 - 80x + 100 = 0$

By calc $x = 5$ or $x = \frac{5}{3}$

$x = 5$, $V = 4(5)(5-5)^2 = 0$

$x = \frac{5}{3}$, $V = 4\left(\frac{5}{3}\right)\left(5 - \frac{5}{3}\right)^2$

$V = \frac{2000}{27} = 74.1$ cm^3 to 3 s.f.

c) $\frac{d^2V}{dx^2} = 24x - 80$, $x = \frac{5}{3}$, $\frac{d^2y}{dx^2} = 24\left(\frac{5}{3}\right) - 80 = -40 < 0$

$\therefore$ a maximum

Leave blank

8.

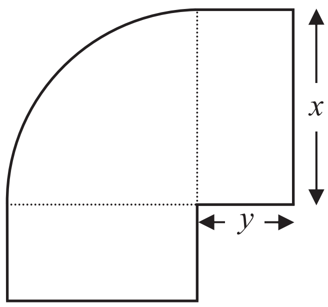

Figure 3

Figure 3 shows a flowerbed. Its shape is a quarter of a circle of radius x metres with two equal rectangles attached to it along its radii. Each rectangle has length equal to x metres and width equal to y metres.

Given that the area of the flowerbed is 4 m^2,

(a) show that

$$y = \frac{16 - \pi x^2}{8x}$$

(3)

(b) Hence show that the perimeter P metres of the flowerbed is given by the equation

$$P = \frac{8}{x} + 2x$$

(3)

(c) Use calculus to find the minimum value of P.

(5)

(d) Find the width of each rectangle when the perimeter is a minimum.
Give your answer to the nearest centimetre.

(2)

a) Area $= xy + xy + \frac{\pi x^2}{4} = 4\text{ m}^2$

$2xy = 4 - \frac{\pi x^2}{4}$

$y = \frac{4 - \frac{\pi x^2}{4}}{2x} = \frac{16 - \pi x^2}{8x}$

P 4 0 0 8 3 A 0 2 2 2 8

b) $$\text{Perimeter} = 4y + 2x + \frac{2\pi x}{4}$$

$$= 4\left(\frac{16 - \pi x^2}{8x}\right) + 2x + \frac{2\pi x}{4}$$

$$= \frac{16 - \pi x^2}{2x} + 2x + \frac{2\pi x}{4}$$

$$= \frac{8}{x} - \cancel{\frac{\pi x}{2}} + 2x + \cancel{\frac{\pi x}{2}}$$

$$\text{Perimeter} = \frac{8}{x} + 2x$$

---

c) $$P = 8x^{-1} + 2x$$

$$\frac{dP}{dx} = -8x^{-2} + 2$$

Min when $\frac{dP}{dx} = 0 \Rightarrow -\frac{8}{x^2} + 2 = 0$

$$2 = \frac{8}{x^2}$$

$$2x^2 = 8$$

$$x^2 = 4$$

$$\underline{x = 2}$$

Min value of $P = \frac{8}{2} + 2(2) = 8\text{ m}$

Check this is a minimum

$$\frac{d^2P}{dx^2} = 16x^{-3} = \frac{16}{x^3} > 0 \text{ for } x = 8 \therefore \text{min}$$

d) $$y = \frac{16 - \pi x^2}{8x}$$

when $x = 2$ $$y = \frac{16 - 4\pi}{16}$$

$$y = 0.2146 \text{ cm}$$

$$y = 21 \text{ cm} \quad \text{to nearest cm}$$

Leave blank

9. A solid glass cylinder, which is used in an expensive laser amplifier, has a volume of 75π cm^3.
The cost of polishing the surface area of this glass cylinder is £2 per cm^2 for the curved surface area and £3 per cm^2 for the circular top and base areas.

Given that the radius of the cylinder is r cm,

(a) show that the cost of the polishing, £C, is given by

$$C = 6\pi r^2 + \frac{300\pi}{r}$$

(4)

(b) Use calculus to find the minimum cost of the polishing, giving your answer to the nearest pound.

(5)

(c) Justify that the answer that you have obtained in part (b) is a minimum.

(1)

a) $V = \pi r^2 h = 75\pi$

$\Rightarrow h = \frac{75\pi}{\pi r^2} = \frac{75}{r^2}$

Curved surface area $= 2\pi r h$

$= 2\pi r \times \frac{75}{r^2}$

$= \frac{150\pi}{r}$

Top and Bottom Area $= \pi r^2 + \pi r^2 = 2\pi r^2$

Cost of polishing $C = 2\left(\frac{150\pi}{r}\right) + 3(2\pi r^2)$

$C = \frac{300\pi}{r} + 6\pi r^2$

b) $$C = 300\pi r^{-1} + 6\pi r^2$$

$$\frac{dC}{dr} = -300\pi r^{-2} + 12\pi r$$

Min when $\frac{dC}{dr} = 0 \Rightarrow -\frac{300\pi}{r^2} + 12\pi r = 0$

$$\Rightarrow \frac{300\pi}{r^2} = 12\pi r$$

$$300\pi = 12\pi r^3$$

$$\frac{300\pi}{12\pi} = r^3$$

$$25 = r^3$$

$$r = \sqrt[3]{25}$$

$$r = 2.924 \text{ cm}$$

Min Cost $= \frac{300\pi}{2.924} + 6\pi \times 2.924^2$

$= £483.48$

$= £483$ to nearest £

---

c) $$\frac{d^2C}{dr^2} = 600\pi r^{-3} + 12\pi$$

$= \frac{600\pi}{r^3} + 12\pi > 0$ for all $r > 0$

$\therefore$ a min at $r = 2.924$ cm

Leave blank

9.

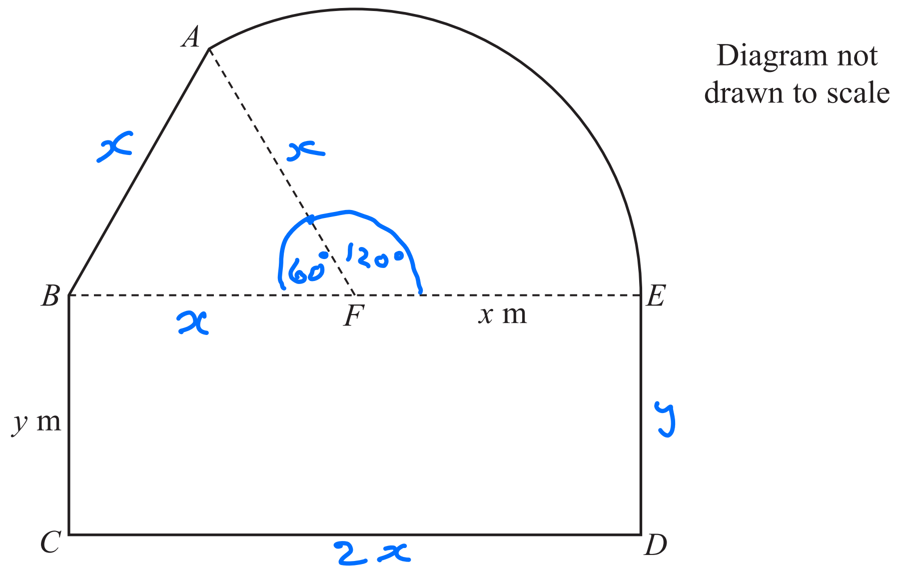

Figure 4

Figure 4 shows a plan view of a sheep enclosure.

The enclosure *ABCDEA*, as shown in Figure 4, consists of a rectangle *BCDE* joined to an equilateral triangle *BFA* and a sector *FEA* of a circle with radius x metres and centre *F*.

The points *B*, *F* and *E* lie on a straight line with $FE = x$ metres and $10 \leqslant x \leqslant 25$

(a) Find, in m^2, the exact area of the sector *FEA*, giving your answer in terms of x, in its simplest form.
(2)

Given that $BC = y$ metres, where $y > 0$, and the area of the enclosure is 1000 m^2,

(b) show that

$$y = \frac{500}{x} - \frac{x}{24}\left(4\pi + 3\sqrt{3}\right)$$

(3)

(c) Hence show that the perimeter P metres of the enclosure is given by

$$P = \frac{1000}{x} + \frac{x}{12}\left(4\pi + 36 - 3\sqrt{3}\right)$$

(3)

(d) Use calculus to find the minimum value of P, giving your answer to the nearest metre.
(5)

(e) Justify, by further differentiation, that the value of P you have found is a minimum.
(2)

a) Area of Sector $= \dfrac{\pi x^2}{3}$

b) whole shape

$$\text{Area} = 2xy + \frac{\pi x^2}{3} + \frac{1}{2}x.x\sin 60^\circ$$

$$1000 = 2xy + \frac{\pi x^2}{3} + \frac{\sqrt{3}}{4}x^2$$

$$1000 - \frac{\pi x^2}{3} - \frac{\sqrt{3}}{4}x^2 = 2xy$$

$$\frac{1000}{2x} - \frac{\pi x^2}{6x} - \frac{\sqrt{3}x^2}{8x} = y$$

$$y = \frac{500}{x} - \frac{\pi x}{6} - \frac{\sqrt{3}x}{8}$$

$$y = \frac{500}{x} - \frac{x}{24}\left(4\pi - 3\sqrt{3}\right)$$

---

c) $$P = 2x + 2y + \frac{2\pi x}{3} + x$$

$$P = 2x + 2\left(\frac{500}{x} - \frac{x}{24}\left(4\pi + 3\sqrt{3}\right)\right) + \frac{2\pi x}{3} + x$$

$$P = 3x + \frac{2\pi x}{3} + \frac{1000}{x} - \frac{x}{12}\left(4\pi + 3\sqrt{3}\right)$$

$$P = \frac{1000}{x} + \frac{36x}{12} + \frac{8\pi x}{12} - \frac{4\pi x}{12} - \frac{3\sqrt{3}x}{12}$$

$$P = \frac{1000}{x} + \frac{x}{12}\left[36 + 4\pi - 3\sqrt{3}\right]$$

d) $P = 1000x^{-1} + \frac{x}{12}\left[36 + 4\pi - 3\sqrt{3}\right]$

$$\frac{dP}{dx} = -1000x^{-2} + \frac{(36 + 4\pi - 3\sqrt{3})}{12}$$

$$\frac{dP}{dx} = -\frac{1000}{x^2} + \frac{(36 + 4\pi - 3\sqrt{3})}{12}$$

At st pt $\frac{dP}{dx} = 0$

$$\frac{1000}{x^2} = \frac{36 + 4\pi - 3\sqrt{3}}{12}$$

$$\frac{x^2}{1000} = \frac{12}{36 + 4\pi - 3\sqrt{3}}$$

$$x^2 = \frac{12000}{36 + 4\pi - 3\sqrt{3}}$$

$$x = \sqrt{\frac{12000}{(36 + 4\pi - 3\sqrt{3})}}$$

$$x = 16.63392808$$

$$\underline{x = 16.6 \text{ m}}$$

$$P = \frac{1000}{16.63392808} + \frac{16.63392808}{12}(4\pi + 36 - 3\sqrt{3})$$

$$P = 120.236 \text{ m}$$

$\underline{\text{Min } P = 120 \text{ m to nearest m}}$

e) $$\frac{d^2A}{dx^2} = -2(-1000x^{-3}) = \frac{2000}{x^3}$$

when $x = 16.6$, $$\frac{d^2A}{dx^2} = \frac{2000}{16.6^3} > 0$$

$\therefore$ a minimum at $x = 16.6$ m