

Leave  
blank

1. Solve the equation

$$7 \operatorname{sech} x - \tanh x = 5$$

Give your answers in the form  $\ln a$  where  $a$  is a rational number.

(5)

$$\frac{7}{\cosh x} - \frac{\sinh x}{\cosh x} = 5$$

$$7 - \sinh x = 5 \cosh x$$

$$5 \cosh x + \sinh x = 7$$

$$\frac{5}{2}(e^x + e^{-x}) + \frac{1}{2}(e^x - e^{-x}) = 7$$

$$5e^x + 5e^{-x} + e^x - e^{-x} = 14$$

$$6e^x + 4e^{-x} = 14$$

$$3e^x + 2e^{-x} = 7$$

$$3e^{2x} - 7e^x + 2 = 0$$

$$(3e^x - 1)(e^x - 2) = 0$$

$$\Rightarrow e^x = \frac{1}{3} \quad \text{or} \quad e^x = 2$$

$$\Rightarrow x = \ln\left(\frac{1}{3}\right) \quad \text{or} \quad x = \ln 2$$



4. Given that  $y = \operatorname{arsinh}(\sqrt{x})$ ,  $x > 0$ ,

(a) find  $\frac{dy}{dx}$ , giving your answer as a simplified fraction. (3)

(b) Hence, or otherwise, find

$$\int_{\frac{1}{4}}^4 \frac{1}{\sqrt{x(x+1)}} dx,$$

giving your answer in the form  $\ln\left(\frac{a+b\sqrt{5}}{2}\right)$ , where  $a$  and  $b$  are integers. (6)

a)  $y = \operatorname{arsinh}(\sqrt{x})$

Let  $u = \sqrt{x}$

$$y = \operatorname{arsinh}(u)$$

$$\frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}}$$

$$\frac{dy}{du} = \frac{1}{\sqrt{1+u^2}}$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{1}{\sqrt{1+u^2}} \times \frac{1}{2\sqrt{x}}$$

$$= \frac{1}{2\sqrt{x+x^2}} = \frac{1}{2\sqrt{x(1+x)}}$$

b)

$$\int_{\frac{1}{4}}^4 \frac{1}{\sqrt{x(x+1)}} dx = 2 \int_{\frac{1}{4}}^4 \frac{1}{2\sqrt{x(1+x)}} dx$$

$$= 2 \left[ \operatorname{arsinh}(\sqrt{x}) \right]_{\frac{1}{4}}^4$$

$$= 2 \left[ \operatorname{arsinh} 2 - \operatorname{arsinh} \left( \frac{1}{2} \right) \right]$$

$$= 2 \left[ \ln(2 + \sqrt{5}) - \ln \left( \frac{1}{2} + \frac{\sqrt{5}}{2} \right) \right]$$



$$\begin{aligned}
&= 2 \ln \left[ \frac{2 + \sqrt{5}}{\frac{1}{2} + \frac{\sqrt{5}}{2}} \right] \\
&= 2 \ln \left[ \frac{4 + 2\sqrt{5}}{1 + \sqrt{5}} \right] \\
&= 2 \ln \left[ \frac{(4 + 2\sqrt{5})(1 - \sqrt{5})}{(1 + \sqrt{5})(1 - \sqrt{5})} \right] \\
&= 2 \ln \left[ \frac{4 + 2\sqrt{5} - 4\sqrt{5} - 10}{1 - 5} \right] \\
&= 2 \ln \left[ \frac{-6 - 2\sqrt{5}}{-4} \right] \\
&= 2 \ln \left[ \frac{3 + \sqrt{5}}{2} \right] \\
&= \ln \left[ \left( \frac{3 + \sqrt{5}}{2} \right)^2 \right] \\
&= \ln \left[ \frac{9 + 6\sqrt{5} + 5}{4} \right] \\
&= \ln \left[ \frac{14 + 6\sqrt{5}}{4} \right] \\
&= \ln \left[ \frac{7 + 3\sqrt{5}}{2} \right]
\end{aligned}$$


---

Leave  
blank

3. (a) Starting from the definitions of  $\sinh x$  and  $\cosh x$  in terms of exponentials, prove that

$$\cosh 2x = 1 + 2 \sinh^2 x \quad (3)$$

- (b) Solve the equation

$$\cosh 2x - 3 \sinh x = 15,$$

giving your answers as exact logarithms. (5)

$$\begin{aligned}
 a) \quad 1 + 2 \sinh^2 x &= 1 + 2 \left[ \frac{1}{2}(e^x - e^{-x}) \right]^2 \\
 &= 1 + 2 \left[ \frac{1}{4} (e^x - e^{-x})^2 \right] \\
 &= 1 + \frac{1}{2} (e^{2x} - 2 + e^{-2x}) \\
 &= 1 + \frac{1}{2} (e^{2x} + e^{-2x}) - 1 \\
 &= \cosh 2x
 \end{aligned}$$

b)  $\cosh 2x - 3 \sinh x = 15$

$$1 + 2 \sinh^2 x - 3 \sinh x = 15$$

$$2 \sinh^2 x - 3 \sinh x - 14 = 0$$

$$(2 \sinh x - 7)(\sinh x + 2) = 0$$

$$\sinh x = \frac{7}{2} \quad \text{or} \quad \sinh x = -2$$

$$x = \operatorname{arsinh}\left(\frac{7}{2}\right) \quad \text{or} \quad x = \operatorname{arsinh}(-2)$$

$$x = \ln\left(\frac{7}{2} + \sqrt{1 + \frac{49}{4}}\right) \quad \text{or} \quad x = \ln(-2 + \sqrt{5})$$

$$x = \ln\left(\frac{7 + \sqrt{53}}{2}\right) \quad \text{or} \quad x = \ln(-2 + \sqrt{5})$$



Leave  
blank

5. Given that  $y = (\operatorname{arcosh} 3x)^2$ , where  $3x > 1$ , show that

$$(a) (9x^2 - 1) \left( \frac{dy}{dx} \right)^2 = 36y, \quad (5)$$

$$(b) (9x^2 - 1) \frac{d^2y}{dx^2} + 9x \frac{dy}{dx} = 18. \quad (4)$$

$$a) \frac{dy}{dx} = 2(\operatorname{arcosh} 3x) \times \frac{1}{\sqrt{(3x)^2 - 1}} \times 3$$

$$\frac{dy}{dx} = 6 \operatorname{arcosh} 3x \times \frac{1}{\sqrt{9x^2 - 1}}$$

$$\left( \frac{dy}{dx} \right)^2 = \frac{36(\operatorname{arcosh} 3x)^2}{9x^2 - 1}$$

$$(9x^2 - 1) \left( \frac{dy}{dx} \right)^2 = 36y$$

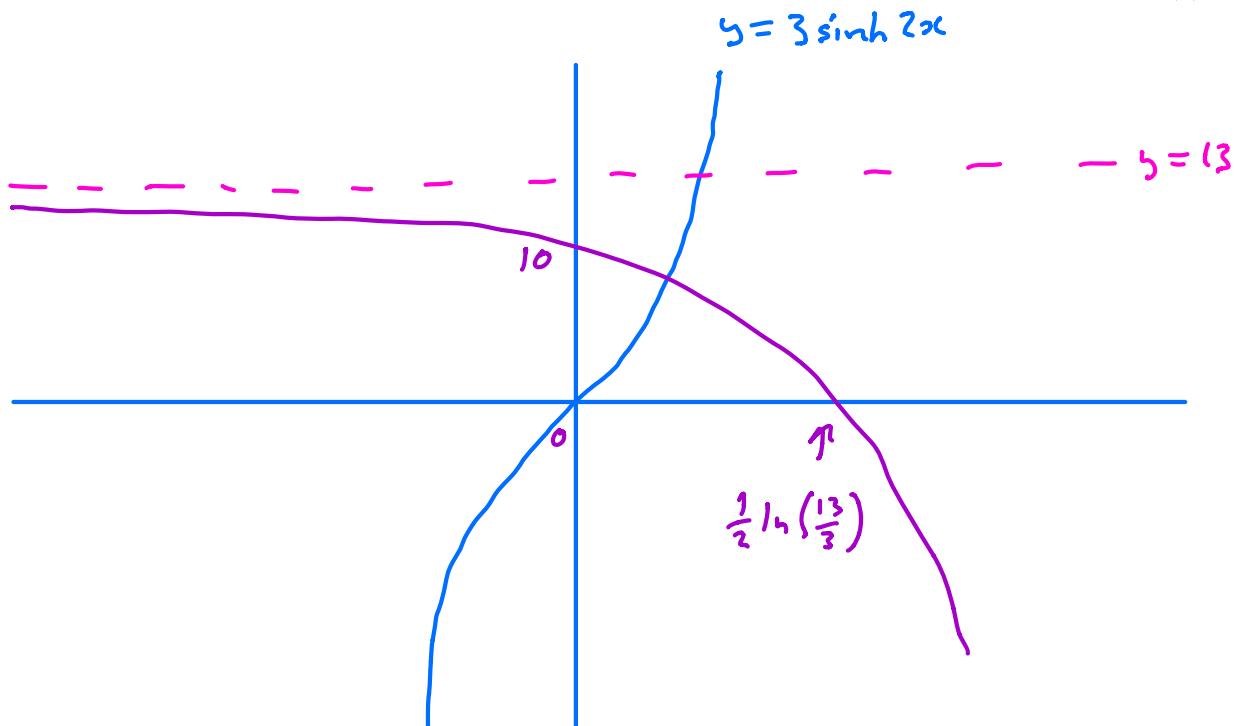
$$b) (9x^2 - 1)_x 2 \left( \frac{dy}{dx} \right) \left( \frac{d^2y}{dx^2} \right) + \left( \frac{dy}{dx} \right)^2 \times 18x = 36 \frac{dy}{dx}$$

$$2(9x^2 - 1) \left( \frac{d^2y}{dx^2} \right) + 18x \left( \frac{dy}{dx} \right) = 36$$

$$(9x^2 - 1) \frac{d^2y}{dx^2} + 9x \frac{dy}{dx} = 18$$



5. The curve  $C_1$  has equation  $y = 3 \sinh 2x$ , and the curve  $C_2$  has equation  $y = 13 - 3e^{2x}$ .
- (a) Sketch the graph of the curves  $C_1$  and  $C_2$  on one set of axes, giving the equation of any asymptote and the coordinates of points where the curves cross the axes. (4)
- (b) Solve the equation  $3 \sinh 2x = 13 - 3e^{2x}$ , giving your answer in the form  $\frac{1}{2} \ln k$ , where  $k$  is an integer. (5)



b)

$$3 \sinh 2x = 13 - 3e^{2x}$$

$$\frac{3}{2} (e^{2x} - e^{-2x}) = 13 - 3e^{2x}$$

$$3e^{2x} - 3e^{-2x} = 26 - 6e^{2x}$$

$$9e^{2x} - 3e^{-2x} - 26 = 0$$

$$9e^{4x} - 26e^{2x} - 3 = 0$$

$$(9e^{2x} + 1)(e^{2x} - 3) = 0$$

$$\Rightarrow e^{2x} = 3$$



Leave  
blank**Question 5 continued**

$$2x = \ln 3$$

$$x = \frac{1}{2} \ln 3$$



P 3 5 4 1 4 A 0 1 3 2 8

5. (a) Differentiate  $x \operatorname{arsinh} 2x$  with respect to  $x$ .

(3)

- (b) Hence, or otherwise, find the exact value of

$$\int_0^{\sqrt{2}} \operatorname{arsinh} 2x \, dx$$

giving your answer in the form  $A \ln B + C$ , where  $A$ ,  $B$  and  $C$  are real.

(7)

$$\begin{aligned} a) \frac{d}{dx} x \operatorname{arsinh} 2x &= x \cdot \frac{1}{\sqrt{1+(2x)^2}} \times 2 + 1_x \operatorname{arsinh} 2x \\ &= \frac{2x}{\sqrt{1+4x^2}} + \operatorname{arsinh} 2x \end{aligned}$$

$$\begin{aligned} b) \int_0^{\sqrt{2}} \left( \frac{2x}{\sqrt{1+4x^2}} + \operatorname{arsinh} 2x \right) dx &= \left[ x \operatorname{arsinh} 2x \right]_0^{\sqrt{2}} \\ \int_0^{\sqrt{2}} \operatorname{arsinh} 2x \, dx &= \left[ x \operatorname{arsinh} 2x \right]_0^{\sqrt{2}} - \int_0^{\sqrt{2}} \frac{2x}{\sqrt{1+4x^2}} \, dx \end{aligned}$$

Aside

$$\int_0^{\sqrt{2}} \frac{2x}{\sqrt{1+4x^2}} \, dx$$

Let  $u = 1+4x^2$

$$\frac{du}{dx} = 8x$$

$$\frac{1}{4} du = 2x \, dx$$

$$\begin{aligned} &= \int_1^9 \frac{1}{4\sqrt{u}} \, du \\ &= \left[ \frac{u^{-\frac{1}{2}}}{2} \right]_1^9 \\ &= \frac{3}{2} - \frac{1}{2} = 1 \end{aligned}$$

$x = \sqrt{2} \quad u = 9$   
 $x = 0 \quad u = 1$



Leave  
blank**Question 5 continued**

$$\int_0^{\sqrt{2}} \operatorname{arsinh} 2x \, dx = \sqrt{2} \operatorname{arsinh}(2\sqrt{2}) - 0 - 1$$
$$= \sqrt{2} \ln \left[ 2\sqrt{2} + \sqrt{(2\sqrt{2})^2 + 1} \right] - 1$$
$$= \sqrt{2} \ln \left[ 2\sqrt{2} + 3 \right] - 1$$

---



Leave  
blank

7.

$$f(x) = 5 \cosh x - 4 \sinh x, \quad x \in \mathbb{R}$$

(a) Show that  $f(x) = \frac{1}{2}(e^x + 9e^{-x})$  (2)

Hence

(b) solve  $f(x) = 5$  (4)

(c) show that  $\int_{\frac{1}{2}\ln 3}^{\ln 3} \frac{1}{5 \cosh x - 4 \sinh x} dx = \frac{\pi}{18}$  (5)

a)  $f(x) = \frac{5}{2}(e^x + e^{-x}) - \frac{4}{2}(e^x - e^{-x})$

$$= \frac{1}{2}e^x + \frac{9}{2}e^{-x}$$

$$= \frac{1}{2}(e^x + 9e^{-x})$$

b)  $f(x) = \frac{1}{2}(e^x + 9e^{-x}) = 5$

$$e^x + 9e^{-x} = 10$$

$$e^{2x} - 10e^x + 9 = 0$$

$$(e^x - 1)(e^x - 9) = 0$$

$$e^x = 1 \quad \text{or} \quad e^x = 9$$

$$x = 0 \quad \text{or} \quad x = \ln 9$$

c)  $\int_{\frac{1}{2}\ln 3}^{\ln 3} \frac{1}{5 \cosh x - 4 \sinh x} dx = \int_{\frac{1}{2}\ln 3}^{\ln 3} \frac{2}{e^x + 9e^{-x}} dx$



Leave  
blank

Question 7 continued

$$= \int_{\frac{1}{2}\ln 3}^{\ln 3} \frac{2e^x}{e^{2x} + 9} dx$$

$$\text{Let } u = e^x$$

$$= \int_{\sqrt{3}}^3 \frac{2}{x^2 + 9} dx$$

$$\frac{du}{dx} = e^x$$

$$du = e^x dx$$

$$= 2 \int_{\sqrt{3}}^3 \frac{1}{x^2 + 3^2} dx$$

$$x = \ln 3 \quad u = 3$$

$$x = \frac{1}{2}\ln 3 \quad u = \sqrt{3}$$

$$= 2 \left[ \frac{1}{3} \tan^{-1} \left( \frac{x}{3} \right) \right]_{\sqrt{3}}^3$$

$$= \frac{2}{3} \left[ \tan^{-1} \left( \frac{3}{3} \right) - \tan^{-1} \left( \frac{\sqrt{3}}{3} \right) \right]$$

$$= \frac{2}{3} \left[ \frac{\pi}{4} - \frac{\pi}{6} \right]$$

$$= \frac{2}{3} \times \frac{\pi}{12}$$

$$= \frac{\pi}{18}$$

