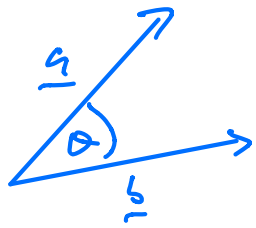
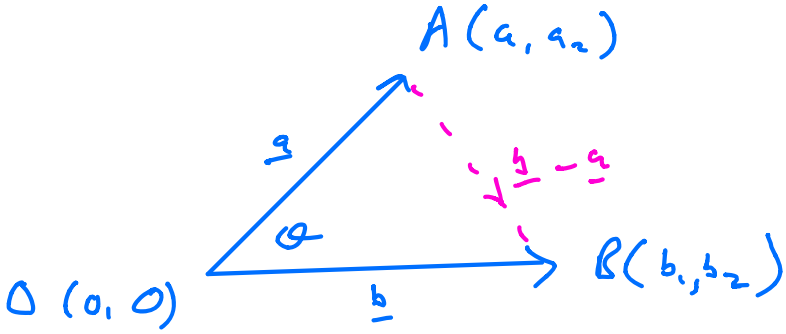
Angle between Vectors



In two dimensions



Cosine Rule

$$\cos\theta = \frac{|\underline{a}|^2 + |\underline{b}|^2 - |\underline{b}-\underline{a}|^2}{2|\underline{a}||\underline{b}|}$$

$$\cos\theta = \frac{a_1^2 + a_2^2 + b_1^2 + b_2^2 - \left((b_1-a_1)^2 + (b_2-a_2)^2\right)}{2|\underline{a}||\underline{b}|}$$

$$\cos\theta = \frac{a_1^2 + a_2^2 + b_1^2 + b_2^2 - \left(b_1^2 + a_1^2 - 2a_1b_1 + b_2^2 + a_2^2 - 2a_2b_2\right)}{2|\underline{a}||\underline{b}|}$$

$$\cos\theta = \frac{2a_1b_1 + 2a_2b_2}{2|\underline{a}||\underline{b}|} = \frac{a_1b_1 + a_2b_2}{|\underline{a}||\underline{b}|}$$

Definition The scalar product of two vectors

$$\underline{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \quad \text{and} \quad \underline{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

is given by

$$\underline{a} \cdot \underline{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

$$\cos\theta = \frac{\underline{a} \cdot \underline{b}}{|\underline{a}||\underline{b}|}$$

<u>Aside</u>

The scalar product (or dotproduct) of $\underline{a}$ and $\underline{b}$ $= |\underline{a}||\underline{b}|\cos\theta$

where θ is angle between them

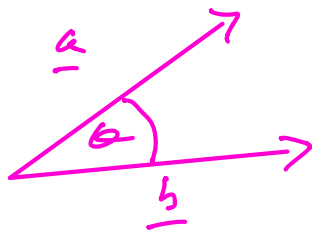
The vector product (or cross product) of $\underline{a}$ and $\underline{b}$ is a vector given by

$$\underline{a} \times \underline{b} = |\underline{a}||\underline{b}|\sin\theta \, \underline{n}$$

where $\underline{n}$ is a unit vector perpendicular to both $\underline{a}$ and $\underline{b}$

VECTOR PRODUCT NOT ON SYLLABUS

Example



$$\underline{a} = \begin{pmatrix} 2 \\ 3 \\ -5 \end{pmatrix}$$

$$\underline{b} = \begin{pmatrix} 4 \\ -1 \\ 6 \end{pmatrix}$$

Find θ

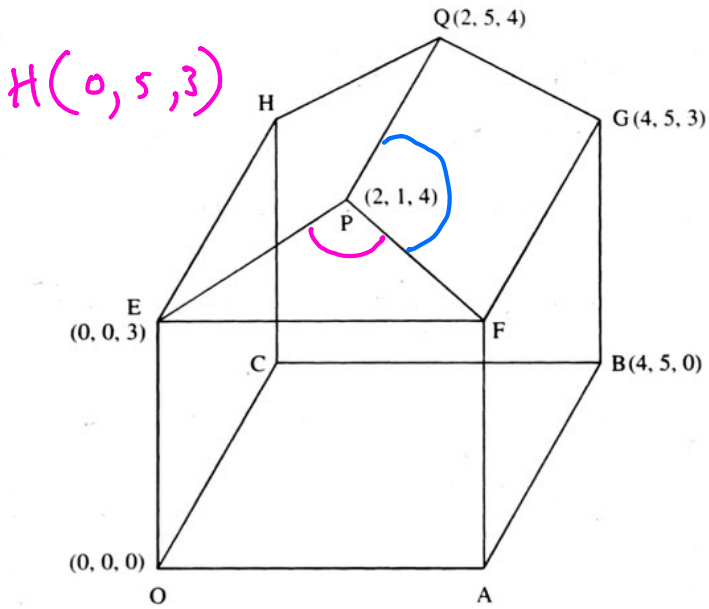
$$\cos\theta = \frac{\underline{a} \cdot \underline{b}}{|\underline{a}||\underline{b}|}$$

$$\cos\theta = \frac{\begin{pmatrix} 2 \\ 3 \\ -5 \end{pmatrix}\begin{pmatrix} 4 \\ -1 \\ 6 \end{pmatrix}}{\left|\begin{pmatrix} 2 \\ 3 \\ -5 \end{pmatrix}\right| \left|\begin{pmatrix} 4 \\ -1 \\ 6 \end{pmatrix}\right|}$$

$$\cos\theta = \frac{8 - 3 - 30}{\sqrt{2^2+3^2+(-5)^2}\sqrt{4^2+(-1)^2+6^2}} = \frac{-25}{\sqrt{38}\sqrt{53}}$$

$$\theta = \cos^{-1}\left(\frac{-25}{\sqrt{38}\sqrt{53}}\right) = 123.9^\circ$$

7 The diagram shows an extension to a house. Its base and walls are rectangular and the end of its roof, EPF, is sloping, as illustrated.



(i) Write down the co-ordinates of A and F.

A(4, 0, 0) F(4, 0, 3)

(ii) Find, using vector methods, the angles FPQ and EPF.

The owner decorates the room with two streamers which are pulled taut. One goes from O to G, the other from A to H. She says that they touch each other and that they are perpendicular to each other.

(iii) Is she right?

ii) E(0, 0, 3)
P(2, 1, 4)
F(4, 0, 3)
Q(2, 5, 4)

$$\overrightarrow{PQ} = \begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix}$$

$$\overrightarrow{PF} = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$$

$$\cos(\angle FPQ) = \frac{\begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}}{\left|\begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix}\right| \left|\begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}\right|} = \frac{0 - 4 - 0}{4\sqrt{4+1+1}}$$

$$= \frac{-4}{4\sqrt{6}}$$

$\angle FPQ = 114.1^\circ$

ii) $\overrightarrow{PE} = \begin{pmatrix} -2 \\ -1 \\ -1 \end{pmatrix}$ $\overrightarrow{PF} = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$

$$\cos(\angle EPF) = \frac{\begin{pmatrix} -2 \\ -1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}}{\left|\begin{pmatrix} -2 \\ -1 \\ -1 \end{pmatrix}\right| \left|\begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}\right|}$$

$$= \frac{-4+1+1}{\sqrt{6}\sqrt{6}} = -\frac{2}{6}$$

$$\angle EPF = \cos^{-1}\left(-\frac{1}{3}\right) = 109.5^\circ$$

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iii) Streamers would meet at $(2, 2.5, 1.5)$

$\overrightarrow{OG} = \begin{pmatrix} 4 \\ 5 \\ 3 \end{pmatrix}$ $\overrightarrow{AH} = \begin{pmatrix} -4 \\ 5 \\ 3 \end{pmatrix}$

$$\overrightarrow{OG} \cdot \overrightarrow{AH} = \begin{pmatrix} 4 \\ 5 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} -4 \\ 5 \\ 3 \end{pmatrix} = -16+25+9$$

$$= 18 \neq 0$$

$\therefore$ not $\perp$

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Q 5 i) iii)

Q 6

Q 8