

## Differentiation Assessment

- 1 Prove, from first principles, that the derivative of  $5x^3$  is  $15x^2$ . (4 marks)

$$\text{Let } f(x) = 5x^3$$

$$\Rightarrow f(x+h) = 5(x+h)^3 \\ = 5(x^3 + 3x^2h + 3xh^2 + h^3)$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{\cancel{5x^3} + 15x^2h + 15xh^2 + 5h^3 - \cancel{5x^3}}{h}$$

$$= \lim_{h \rightarrow 0} (15x^2 + 15xh + 5h)$$

$$= 15x^2 + 0 + 0$$

$$f'(x) = 15x^2$$


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2  $f(x) = x^3 - 4x^2 - 35x + 20$

Find the set of values of  $x$  for which  $f(x)$  is increasing.

(5 marks)

$f(x)$  is increasing when  $f'(x) > 0$

$$f'(x) = 3x^2 - 8x - 35$$

If  $f'(x) > 0$

$$3x^2 - 8x - 35 > 0$$

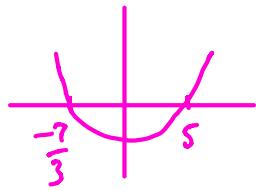
$$\begin{aligned} 3x - 35 \\ = -105 \\ +7 -15 \end{aligned}$$

$$3x^2 + 7x - 15x - 35 > 0$$

$$x(3x+7) - 5(3x+7) > 0$$

$$y = 3x^2 - 8x - 35$$

$$(x-5)(3x+7) > 0$$



Either  $x < -\frac{7}{3}$  or  $x > 5$

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- 3 A curve  $C$  has equation  $y = x^3 - x^2 - x + 2$

The point  $P$  has  $x$ -coordinate 2

a Find  $\frac{dy}{dx}$  in terms of  $x$ . (2 marks)

b Find the equation of the tangent to the curve  $C$  at the point  $P$ . (4 marks)

The normal to  $C$  at  $P$  intersects the  $x$ -axis at  $A$ .

c Find the coordinates of  $A$ . (4 marks)

a)  $\frac{dy}{dx} = 3x^2 - 2x - 1$

b) When  $x = 2$ ,  $y = 2^3 - 2^2 - 2 + 2 = 4$   
 $\therefore P(2, 4)$

When  $x = 2$ ,  $\frac{dy}{dx} = 3(2)^2 - 2(2) - 1 = 7$

Tangent at  $P$

$$y - y_1 = m(x - x_1)$$

$$y - 4 = 7(x - 2)$$

$$y - 4 = 7x - 14$$

$$\underline{y = 7x - 10}$$

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c) Normal has gradient  $= -\frac{1}{7}$

$$y - y_1 = m(x - x_1)$$

$$y - 4 = -\frac{1}{7}(x - 2)$$

$$7y - 28 = -x + 2$$

$$\text{Normal } x + 7y - 30 = 0$$

Crosses  $x$ -axis when  $y = 0$

$$x + 0 - 30 = 0$$

$$\underline{x = 30}$$

$$\therefore A(30, 0)$$


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4  $f(x) = x^3 - 7x^2 - 24x + 18$

Sketch the graph of the gradient function,  $y = f'(x)$ . Use algebraic methods to determine any points where the graph cuts the coordinate axes and mark these on the graph. Using calculus, find the coordinates of any turning points on the graph. (9 marks)

$$f'(x) = 3x^2 - 14x - 24$$

Cuts  $x$ -axis when  $3x^2 - 14x - 24 = 0$

$$\begin{array}{r} 3x^2 - 24 \\ = -72 \\ +4 \quad -18 \end{array}$$

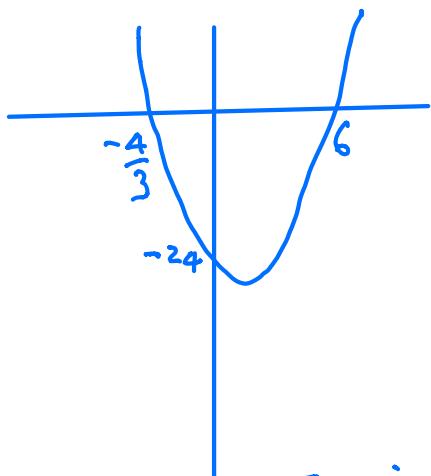
$$3x^2 + 4x - 18x - 24 = 0$$

$$x(3x+4) - 6(3x+4) = 0$$

$$(x-6)(3x+4) = 0$$

$$x = 6 \text{ or } x = -\frac{4}{3}$$

Cuts  $y$ -axis at  $-24$



$$f''(x) = 6x - 14$$

graph has E.p. when  $6x - 14 = 0$

$$6x = 14$$

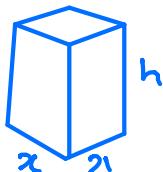
$$x = \frac{14}{6} = \frac{7}{3}$$

$$y = 3\left(\frac{7}{3}\right)^2 - 14\left(\frac{7}{3}\right) - 24 = -\frac{121}{3}$$

Turning point at  $\left(\frac{7}{3}, -\frac{121}{3}\right)$

- 5 A fish tank in the shape of a cuboid is to be made from 1600 cm<sup>2</sup> of glass.  
 The fish tank will have a square base of side length  $x$  cm, and no lid. No glass is wasted.  
 The glass can be assumed to be very thin.
- a Show that the volume,  $V$  cm<sup>3</sup>, of the fish tank is given by  $V = 400x - \frac{x^3}{4}$ . **(5 marks)**
- b Given that  $x$  can vary, use differentiation to find the maximum or minimum value of  $V$ . **(4 marks)**
- c Justify that the value of  $V$  you found in part b is a maximum. **(2 marks)**

a)



$$V = x^2 h$$

$$\text{Area} = 1600 = x^2 + 4xh$$

$$1600 - x^2 = 4xh$$

$$\frac{1600 - x^2}{4x} = h$$

$$\frac{400}{x} - \frac{x}{4} = h$$

Sub for  $h$  in  $V$

$$V = x^2 \left( \frac{400}{x} - \frac{x}{4} \right)$$

$$V = 400x - \frac{x^3}{4}$$


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b)

$$\frac{dV}{dx} = 400 - \frac{3x^2}{4}$$

$$\text{At E.p. } \frac{dV}{dx} = 0 \Rightarrow 400 - \frac{3x^2}{4} = 0$$

$$1600 = 3x^2$$

$$\frac{1600}{3} = x^2$$

$$x = \frac{40}{\sqrt{3}}$$

$$\text{When } x = \frac{40}{\sqrt{3}}$$

$$V = 400 \left( \frac{40}{\sqrt{3}} \right) - \frac{\left( \frac{40}{\sqrt{3}} \right)^3}{4}$$

max or min at  $V = 6158 \text{ cm}^3$

c)  $\frac{d^2 V}{dx^2} = -\frac{6x}{4} = -\frac{3x}{2}$

When  $x = \frac{40}{\sqrt{3}}$   $\frac{d^2 V}{dx^2} = -\frac{3\left(\frac{40}{\sqrt{3}}\right)}{2} < 0$

$\therefore V = 6158 \text{ cm}^3$  is a maximum

- 6 Figure 1 shows the plan of a school running track. It consists of two straight sections, which are the opposite sides of a rectangle, and two semicircular sections, each of radius  $r$  m. The length of the track is 300 m and it can be assumed to be very narrow.

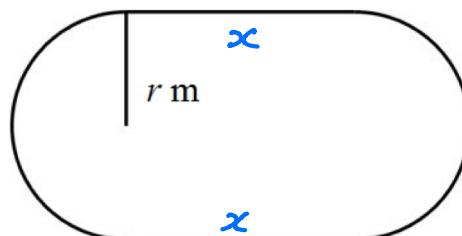


Figure 1

- a Show that the internal area,  $A \text{ m}^2$ , is given by the formula  $A = 300r - \pi r^2$ . **(5 marks)**
- b Hence find in terms of  $\pi$  the maximum value of the internal area.  
You do not have to justify that the value is a maximum. **(6 marks)**

Let straight measure  $x$

$$\begin{aligned}\text{Perimeter} &= 2x + 2\pi r = 300 \\ \Rightarrow x + \pi r &= 150 \\ x &= 150 - \pi r\end{aligned}$$

Area = Area of Circle + Area of Rectangle

$$A = \pi r^2 + 2rx$$

$$A = \pi r^2 + 2r(150 - \pi r)$$

$$A = \pi r^2 + 300r - 2\pi r^2$$

$$A = 300r - \pi r^2$$

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b)

$$\frac{dA}{dr} = 300 - 2\pi r$$

L.P. when  $\frac{dA}{dr} = 0 \Rightarrow 300 - 2\pi r = 0$

$$300 = 2\pi r$$

$$\frac{300}{2\pi} = r$$

$$r = \frac{150}{\pi}$$

Assuming this to give a maximum

$$\text{Max Area} = 300\left(\frac{150}{\pi}\right) - \pi\left(\frac{150}{\pi}\right)^2$$

$$= \frac{45000}{\pi} - \frac{22500}{\pi}$$

$$= \frac{22500}{\pi} \text{ m}^2$$

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