

Tangents to Polar Curves

Review of Parametric Equations

$$\text{If } x = f(\theta), y = g(\theta)$$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

Chain Rule

$$\frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx}$$

Example

$$x = 2\cos\theta, y = 2\sin\theta$$

$$\frac{dx}{d\theta} = -2\sin\theta \quad \frac{dy}{d\theta} = 2\cos\theta$$

$$x^2 + y^2 = 4\cos^2\theta + 4\sin^2\theta$$

$$x^2 + y^2 = 4(\cos^2\theta + \sin^2\theta)$$

$$x^2 + y^2 = 2^2$$

$$\frac{dy}{dx} = \frac{\cancel{\frac{dy}{d\theta}}}{\cancel{\frac{dx}{d\theta}}} = \frac{2\cos\theta}{-2\sin\theta} = -\cot\theta$$

Exercise 5D

i) $r = a(1 + \cos\theta)$

$$x = r\cos\theta = a(\cos\theta + \cos^2\theta)$$

$$y = r\sin\theta = a(\sin\theta + \sin\theta\cos\theta)$$

$$\frac{dy}{dx} = \frac{\cancel{\frac{dy}{d\theta}}}{\cancel{\frac{dx}{d\theta}}}$$

\perp to initial line when $\frac{dy}{dx} = \infty$

$$\Rightarrow \frac{dx}{d\theta} = 0$$

$$x = a \left[\cos\theta + \cos^2\theta \right]$$

$$\frac{dx}{d\theta} = a \left[-\sin\theta + 2\cos\theta(-\sin\theta) \right]$$

$$= a \left[-\sin\theta (1 + 2\cos\theta) \right]$$

$$\frac{dx}{d\theta} = 0 \Rightarrow \sin\theta = 0 \text{ or } 1 + 2\cos\theta = 0$$

$$\theta = 0, \pi, 2\pi \quad \cos\theta = -\frac{1}{2}$$

$$\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$$

Points where tangents \perp to initial line

$$(2a, 0) \quad (\cancel{0, \pi}) \quad \left(\frac{a}{2}, \frac{2\pi}{3}\right) \quad \left(\frac{a}{2}, \frac{4\pi}{3}\right)$$

$$2) \quad r = e^{2\theta} \quad 0 \leq \theta \leq \pi$$

$$y = r \sin\theta = e^{2\theta} \sin\theta$$

$$x = r \cos\theta = e^{2\theta} \cos\theta$$

$$\frac{dx}{d\theta} = e^{2\theta}(-\sin\theta) + 2e^{2\theta}\cos\theta = e^{2\theta}(2\cos\theta - \sin\theta)$$

$$\frac{dy}{d\theta} = e^{2\theta}\cos\theta + 2e^{2\theta}\sin\theta = e^{2\theta}(\cos\theta + 2\sin\theta)$$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{e^{2\theta}(\cos\theta + 2\sin\theta)}{e^{2\theta}(2\cos\theta - \sin\theta)}$$

$$\frac{dy}{dx} = \frac{\cos \theta + 2\sin \theta}{2\cos \theta - \sin \theta}$$

Tangents parallel to initial line

$$\Rightarrow \cos \theta + 2\sin \theta = 0$$

$$\cos \theta = -2\sin \theta$$

$$1 = \frac{-2\sin \theta}{\cos \theta}$$

$$-\frac{1}{2} = \tan \theta$$

$$\theta = \tan^{-1}(-\frac{1}{2})$$

$$\theta = 2.68 \text{ radians}$$

$$r = e^{2 \times 2.68779405045}$$

$$(2.12, 2.68)$$

Tgts \perp to initial line

$$\Rightarrow 2\cos \theta - \sin \theta = 0$$

$$2\cos \theta = \sin \theta$$

$$2 = \frac{\sin \theta}{\cos \theta}$$

$$\tan^{-1} 2 = \theta$$

$$\theta = 1.11 \text{ radians}$$

$$r = e^{2 \times 1.102148718}$$

$$(9.15, 1.11)$$

$$3) r = a \cos 2\theta \quad -\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}$$

$$x = r \cos \theta = a \cos 2\theta \cos \theta$$

$$y = r \sin \theta = a \cos 2\theta \sin \theta$$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

\overline{AO}

tgt parallel to initial line $\Rightarrow \frac{dy}{d\theta} = 0$

$$\frac{dy}{d\theta} = a \cos 2\theta \cos \theta - 2a \sin 2\theta \sin \theta$$

$$\frac{dy}{d\theta} = 0 \Rightarrow a [\cos 2\theta \cos \theta - 2 \sin 2\theta \sin \theta] = 0$$

$$\Rightarrow \cos^2 \theta \cos \theta - 4 \sin^2 \theta \cos \theta = 0$$

$$\cos \theta (\cos 2\theta - 4 \sin^2 \theta) = 0$$

$$\cos \theta (1 - 6 \sin^2 \theta) = 0$$

$\cos \theta = 0$ no solution for $-\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}$

$$\text{or } \sin^2 \theta = \frac{1}{6}$$

$$\sin \theta = \pm \frac{1}{\sqrt{6}} \quad \theta = \pm 0.4205343353$$

$$\theta = \pm 0.421$$

$$r = \frac{2a}{3} \quad r = \frac{2a}{3}$$

$$\left(\frac{2a}{3}, 0.421\right) \quad \left(\frac{2a}{3}, -0.421\right)$$