Partial Fractions

Exercise 3E

Q1a $\dfrac{1}{(x^2+1)(x+3)} \equiv \dfrac{Ax+B}{x^2+1} + \dfrac{C}{x+3}$

$$1 \equiv (Ax+B)(x+3) + C(x^2+1)$$

$x=-3$ $\quad 1 = C((-3)^2+1)$

$1 = 10C \quad \Rightarrow C = \frac{1}{10}$

$x=0$ $\quad 1 = B(3) + \frac{1}{10}(1)$

$\frac{9}{10} = 3B \quad \Rightarrow B = \frac{3}{10}$

coeff of x^2 $\quad 0 = A + C$

$0 = A + \frac{1}{10} \quad \Rightarrow A = -\frac{1}{10}$

$$\frac{1}{(x^2+1)(x+3)} \equiv \frac{3-x}{10(x^2+1)} + \frac{1}{10(x+3)}$$

Q3 $f(x) = x^4 - x^3 - 4x^2 - 2x - 12$

$$\begin{array}{r|l}
 & x^3 - 3x^2 + 2x - 6 \\
\hline
x+2 & x^4 - x^3 - 4x^2 - 2x - 12 \\
 & \underline{x^4 + 2x^3} \\
 & -3x^3 - 4x^2 \\
 & \underline{-3x^3 - 6x^2} \\
 & +2x^2 - 2x \\
 & \underline{+2x^2 + 4x} \\
 & -6x - 12
\end{array}$$

$$-6x - 12$$

Find factors of

$g(x) = x^3 - 3x^2 + 2x - 6$

$g(-1) = -1 - 3 - 2 - 6$

$g(1) = 1 - 3 + 2 - 6$

$g(3) = 27 - 27 + 6 - 6$ ✓ $(x - 3)$ is a factor

$$\begin{array}{r|l} & x^2 + 2 \\ \hline x - 3 & x^3 - 3x^2 + 2x - 6 \\ & \underline{x^3 - 3x^2} \\ & \qquad\qquad + 2x - 6 \\ & \qquad\qquad \underline{+ 2x - 6} \end{array}$$

$$f(x) = (x+2)(x-3)(x^2+2)$$

b) $$\frac{x^3 - 20x^2 + 4x - 24}{x^4 - x^3 - 4x^2 - 2x - 12}$$

$$= \frac{x^3 - 20x^2 + 4x - 24}{(x+2)(x-3)(x^2+2)}$$

$$x^3 - 20x^2 + 4x - 24 \equiv \frac{A}{x+2} + \frac{B}{x-3} + \frac{Cx+D}{x^2+2}$$

$$x^3 - 20x^2 + 4x - 24 \equiv A(x-3)(x^2+2) + B(x+2)(x^2+2) + (Cx+D)(x+2)(x-3)$$

$x = -2$

$$-8 - 80 - 8 - 24 = A(-5)(6)$$

$$-120 = -30A \quad \Rightarrow \quad A = 4$$

$x = 3$

$$27 - 180 + 12 - 24 = B(5)(11)$$
$$-165 = 55B \quad \Rightarrow \quad B = -3$$

$x = 0$

$$-24 = A(-3)(2) + B(2)(2) + D(2)(-3)$$
$$\cancel{-24} = \cancel{-24} - 12 - 6D$$
$$12 = -6D \quad \Rightarrow \quad D = -2$$

Coeff of x^3

$$1 = A + B + C$$
$$1 = 4 - 3 + C$$
$$1 = 1 + C \quad \Rightarrow \quad C = 0$$

$$\int \left(\frac{4}{x+2} - \frac{3}{x-3} - \frac{2}{x^2+2} \right) dx$$
$$= 4\ln(x+2) - 3\ln(x-3) - \frac{2}{\sqrt{2}} \tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + c$$
$$= \ln(x+2)^4 - \ln(x-3)^3 - \sqrt{2}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + c$$
$$= \ln\left(\frac{(x+2)^4}{(x-3)^3}\right) - \sqrt{2}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + c$$

Q5 $$\int \frac{x^2+1}{4x^4+9x^2}\,dx = \int \frac{x^2+1}{x^2(4x^2+9)}\,dx$$

$$\frac{x^2+1}{x^2(4x^2+9)} \equiv \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{4x^2+9}$$

$$x^2+1 \equiv Ax(4x^2+9) + B(4x^2+9) + (Cx+D)x^2$$

$x=0$ $\quad 1 = 9B \quad \Rightarrow B = \frac{1}{9}$

coeff of x^3 $\quad 0 = 4A + C \quad *$

coeff of x^2 $\quad 1 = 4B + D$

$1 = \frac{4}{9} + D \quad \Rightarrow D = \frac{5}{9}$

$\frac{5}{9} = D$

coeff of x $\quad 0 = 9A \quad \Rightarrow A = 0$

$*$ $\quad 0 = 4A + C$

$0 = 0 + C \quad \Rightarrow C = 0$

Integral becomes

$$\int \left(+ \frac{1}{9x^2} + \frac{5}{9(4x^2+9)} \right) dx$$

$$\int \left(\frac{1}{9x^2} + \frac{5}{36\left(x^2+\left(\frac{3}{2}\right)^2\right)} \right) dx$$

$$= -\frac{1}{9x} + \frac{5}{54} \tan^{-1}\left(\frac{2x}{3}\right) + c$$

Hwk Exercise 3E Q2, Q4