

FP2 Paper *adapted 2008

1. Solve the differential equation $\frac{dy}{dx} - 3y = x$

to obtain y as a function of x .

(Total 5 marks)

$$\text{I.F. } e^{\int -3dx} \\ = e^{-3x} \quad e^{-3x} \frac{dy}{dx} - 3e^{-3x}y = xe^{-3x} \\ \frac{d}{dx} (e^{-3x}y) = xe^{-3x}$$
$$e^{-3x}y = \int xe^{-3x}dx$$

$$\int xe^{-3x}dx \quad \text{Let } u = x \quad \text{Let } \frac{du}{dx} = e^{-3x}$$
$$\frac{du}{dx} = 1 \quad v = -\frac{1}{3}e^{-3x}$$

$$\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$$

$$\int xe^{-3x}dx = -\frac{1}{3}xe^{-3x} + \int \frac{1}{3}e^{-3x}dx \\ = -\frac{1}{3}xe^{-3x} - \frac{1}{9}e^{-3x} + C$$

$$e^{-3x}y = -\frac{1}{3}xe^{-3x} - \frac{1}{9}e^{-3x} + C$$

$$y = -\frac{1}{3}x - \frac{1}{9} + Ce^{3x}$$

3. (a) Find the general solution of the differential equation $3\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = x^2$ (8)

Aux Eqn $3m^2 - m - 2 = 0$
 $(3m+2)(m-1) = 0$
 $m = -\frac{2}{3}, m = 1$

C.F. $y = Ae^{-\frac{2}{3}x} + Be^x$

P.I. $y = ax^2 + bx + c$

$$\frac{dy}{dx} = 2ax + b$$

$$\frac{d^2y}{dx^2} = 2a$$

$$3(2a) - (2ax + b) - 2(ax^2 + bx + c) = x^2$$

$$6a - 2ax - b - 2ax^2 - 2bx - 2c = x^2$$

$$6a - b - 2c - 2(a+b)x - 2ax^2 = x^2$$

$$\Rightarrow -2a = 1 \quad a = -\frac{1}{2}$$

$$\Rightarrow b = \frac{1}{2}$$

$$\Rightarrow 6(-\frac{1}{2}) - \frac{1}{2} - 2c = 0$$

$$\Rightarrow -\frac{7}{2} - 2c = 0$$

$$\Rightarrow c = -\frac{7}{4}$$

P.I. $y = -\frac{1}{2}x^2 + \frac{1}{2}x - \frac{7}{4}$

Gen Solution $\underline{y = Ae^{-\frac{2}{3}x} + Be^x - \frac{1}{2}x^2 + \frac{1}{2}x - \frac{7}{4}}$

(b) Find the particular solution for which, at $x = 0$, $y = 2$ and $\frac{dy}{dx} = 3$. (6)(Total 14 marks)

$$\begin{aligned}x &= 0 & 2 &= A + B - \frac{7}{4} \\y &= 2 & \frac{15}{4} &= A + B\end{aligned}\quad (1)$$

$$\frac{dy}{dx} = -\frac{2}{3}Ae^{-\frac{2}{3}x} + Be^x - 2x + \frac{1}{2}$$

$$\begin{aligned}x &= 0 & 3 &= -\frac{2}{3}A + B + \frac{1}{2} \\ \frac{dy}{dx} &= 3 & \frac{5}{2} &= -\frac{2}{3}A + B\end{aligned}\quad (2)$$

$$(1) - (2) \quad \frac{5}{4} = \frac{5}{3}A \quad \Rightarrow \quad A = \frac{3}{4}$$

$$\text{Sub in } (1) \quad \frac{15}{4} = \frac{3}{4} + B \quad \Rightarrow \quad B = 3$$

Particular Solution

$$y = \frac{3}{4}e^{-\frac{2}{3}x} + 3e^x - \frac{1}{2}x^2 + \frac{1}{2}x - \frac{7}{4}$$

5. (a) Find, in terms of k , the general solution of the differential equation

$$\frac{d^2x}{dt^2} + 4 \frac{dx}{dt} + 3x = kt + 5, \text{ where } k \text{ is a constant and } t > 0. \quad (7)$$

For large values of t , this general solution may be approximated by a linear function.

- (b) Given that $k = 6$, find the equation of this linear function. (2)(Total 9 marks)

a) Aux Eqn $m^2 + 4m + 3 = 0$
 $(m+3)(m+1) = 0$
 $m = -3, m = -1$

C.F. $x = Ae^{-3t} + Be^{-t}$

P.I. $x = \lambda t + \mu$

$$\frac{dx}{dt} = \lambda$$

$$\frac{d^2x}{dt^2} = 0$$

$$0 + 4\lambda + 3\lambda t + 3\mu = kt + 5$$

$$\Rightarrow 3\lambda = k \Rightarrow \lambda = \frac{k}{3}$$

$$\Rightarrow 4\left(\frac{k}{3}\right) + 3\mu = 5$$

$$\Rightarrow 4k + 9\mu = 15$$

$$\mu = \frac{15 - 4k}{9}$$

P.I. $x = \frac{k}{3}t + \frac{15 - 4k}{9}$

General Solution

$$x = Ae^{-3t} + Be^{-t} + \frac{k}{3}t + \frac{15 - 4k}{9}$$

b)

$$x \approx \frac{6}{3}t + \frac{15 - 4(6)}{9} \quad x \approx 2t - 1$$

7. (a) Show that the substitution $y = vx$ transforms the differential equation

$$\frac{dy}{dx} = \frac{x}{y} + \frac{3y}{x}, \quad x > 0, \quad y > 0 \quad (\text{I})$$

$$\text{into the differential equation } x \frac{dv}{dx} = 2v + \frac{1}{v}. \quad (\text{II}) \quad (3)$$

a) $\frac{dy}{dx} = \frac{x}{y} + \frac{3y}{x}$

If $y = vx$ $\frac{dy}{dx} = \frac{x}{vx} + \frac{3vx}{x}$

$$\frac{dy}{dx} = \frac{1}{v} + 3v$$

$y = vx$ (differentiate as product of v and x)

$$\frac{dy}{dx} = v + x \frac{dv}{dx} = v + x \frac{dv}{dx}$$

$$\Rightarrow v + x \frac{dv}{dx} = \frac{1}{v} + 3v$$

$$\Rightarrow x \frac{dv}{dx} = \frac{1}{v} + 2v$$

- (b) By solving differential equation (II), find a general solution of differential equation (I) in the form $y = f(x)$. (7)

Given that $y = 3$ at $x = 1$, (c) find the particular solution of differential equation (I). (2)

b) $x \frac{dv}{dx} = \frac{1 + 2v^2}{v}$

$$\int \frac{v}{2v^2+1} dv = \int \frac{1}{x} dx$$

$$\frac{1}{4} \ln(2v^2+1) = \ln x + C$$

$$\frac{1}{4} \ln(2v^2+1) = \ln x + \ln A$$

$$\frac{1}{4} \ln(2v^2+1) = \ln Ax$$

$$\ln(2v^2+1) = 4 \ln Ax$$

$$\ln(2v^2+1) = \ln(Ax)^4$$

$$2v^2+1 = Ax^4 \quad (A \text{ a positive constant})$$

$$v^2 = \frac{Ax^4 - 1}{2}$$

$$v = \sqrt{\frac{Ax^4 - 1}{2}}$$

$$y = vx$$

$$y = x \sqrt{\frac{Ax^4 - 1}{2}}$$

c) $y = 3$ $3 = 1 \sqrt{\frac{A - 1}{2}}$
 $x = 1$

$$9 = \frac{A - 1}{2}$$

$$A = 19$$

$$y = x \sqrt{\frac{19x^4 - 1}{2}}$$

9.

$$(x^2 + 1) \frac{d^2 y}{dx^2} = 2y^2 + (1 - 2x) \frac{dy}{dx} \quad (\text{I})$$

(a) By differentiating equation (I) with respect to x , show that

$$(x^2 + 1) \frac{d^3 y}{dx^3} = (1 - 4x) \frac{d^2 y}{dx^2} + (4y - 2) \frac{dy}{dx}. \quad (3)$$

Given that $y = 1$ and $\frac{dy}{dx} = 1$ at $x = 0$,

(b) find the series solution for y , in ascending powers of x , up to and including the term in x_3 . (4)

(c) Use your series to estimate the value of y at $x = -0.5$, giving your answer to two decimal places. (1)

$$\begin{aligned} a) \quad & (x^2 + 1) \frac{d^3 y}{dx^3} + 2x \frac{d^2 y}{dx^2} = 4y \frac{dy}{dx} + (1 - 2x) \frac{d^2 y}{dx^2} - 2 \frac{dy}{dx} \end{aligned}$$

$$(x^2 + 1) \frac{d^3 y}{dx^3} = (1 - 4x) \frac{d^2 y}{dx^2} + (4y - 2) \frac{dy}{dx}$$

$$b) \quad \text{Let } y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

$$\frac{dy}{dx} = a_1 + 2a_2 x + 3a_3 x^2 + \dots$$

$$\frac{d^2 y}{dx^2} = 2a_2 + 6a_3 x + \dots$$

$$\frac{d^3 y}{dx^3} = 6a_3 + 24a_4 x + \dots$$

$$x = 0, \quad y = 1, \quad \frac{dy}{dx} = 1$$

$$\Rightarrow a_0 = 1 , a_1 = 1$$

Sub for $x=0, \frac{dy}{dx}=1, y=1$ in (1)

$$(0+1) \frac{d^2y}{dx^2} = 2(1)^2 + (1-0)1$$

$$\Rightarrow \frac{d^2y}{dx^2} = 3 \quad \Rightarrow 2a_2 = 3 \\ a_2 = \frac{3}{2}$$

Sub for $x=0, \frac{dy}{dx}=1, y=1, \frac{d^2y}{dx^2}=3$ in (3)

$$(0+1) \frac{d^3y}{dx^3} = (1-0) \frac{d^2y}{dx^2} + (4-2) \frac{dy}{dx}$$

$$\frac{d^3y}{dx^3} = 3 + 2(1) = 5$$

$$\Rightarrow 6a_3 = 5 \quad a_3 = \frac{5}{6}$$

$$y = 1 + x + \frac{3}{2}x^2 + \frac{5}{6}x^3 + \dots$$

c) $x = -0.5 \quad y \approx 1 - 0.5 + \frac{3}{2}(-0.5)^2 + \frac{5}{6}(-0.5)^3$

$$y \approx 0.77$$