AQA FP3 2nd Order Differential Equations Questions

1 a) $m^2 + 2m + 2 = 0$

$$m = \frac{-2 \pm \sqrt{4-8}}{2} = -1 \pm i$$

b) i) $$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 4x$$

CF $$y = e^{-x}(A\cos x + B\sin x)$$

PI $$y = \lambda + \mu x$$

$$\frac{dy}{dx} = \mu$$

$$\frac{d^2y}{dx^2} = 0$$

$$0 + 2\mu + 2(\lambda + \mu x) = 4x$$

$$2\mu + 2\lambda + 2\mu x = 4x$$

$$\Rightarrow \quad \mu = 2, \quad \lambda = -2$$

Gen Soln $$y = e^{-x}(A\cos x + B\sin x) + 2x - 2$$

---

ii) $y = 1$, $\frac{dy}{dx} = 2$, when $x = 0$

$$1 = e^0(A\cos 0 + B\sin 0) + 0 - 2$$

$$\underline{3 = A}$$

$$\frac{dy}{dx} = e^{-x}(-A\sin x + B\cos x)$$
$$- e^{-x}(A\cos x + B\sin x) + 2$$

$$2 = B - A + 2$$
$$0 = B - A \qquad \Rightarrow B = A = 3$$

$$y = 3e^{-x}(\cos x + \sin x) + 2x - 2$$

---

1) $$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 4y = 8x - 10 - 10\cos 2x$$

a) $$y = 2x + \sin 2x$$
$$\frac{dy}{dx} = 2 + 2\cos 2x$$
$$\frac{d^2y}{dx^2} = -4\sin 2x$$

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 4y$$

$$= -4\sin 2x - 5(2 + 2\cos 2x) + 4(2x + \sin 2x)$$

$$= \cancel{-4\sin 2x} - 10 - 10\cos 2x + 8x + \cancel{4\sin 2x}$$

$$= 8x - 10 - 10\cos 2x \qquad \text{as required}$$

$\therefore \quad y = 2x + \sin 2x$ is a particular integral

---

b) Aux Eqn $\quad m^2 - 5m + 4 = 0$

$$(m-1)(m-4) = 0$$

$$m = 1 \quad \text{or} \quad m = 4$$

C.F. $\quad y = Ae^{x} + Be^{4x}$

Gen Sol $\quad$ CF + PI

$$y = Ae^{x} + Be^{4x} + 2x + \sin 2x$$

---

c) $\quad y = 2, \ \frac{dy}{dx} = 0$ when $x = 0$

$x = 0 \Rightarrow \quad 2 = A + B \qquad (1)$

$$\frac{dy}{dx} = Ae^{x} + 4Be^{x} + 2 + 2\cos 2x$$

$x = 0 \Rightarrow \quad 0 = A + 4B + 2 + 2$

$$-4 = A + 4B \qquad (2)$$

② - ① $\quad -6 = 3B \quad \Rightarrow B = -2$

$\Rightarrow A = 4$

$$\therefore \quad y = 4e^{x} - 2e^{4x} + 2x + \sin 2x$$

---

6) a) $\quad u = \dfrac{dy}{dx} + 2y$

diff x $\quad \dfrac{du}{dx} = \dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx}$

$$\Rightarrow \frac{du}{dx} + 2u = \frac{d^2y}{dx^2} + \frac{2dy}{dx} + \frac{2dy}{dx} + 4y$$

$$= \frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y$$

$$\Rightarrow \frac{du}{dx} + 2u = e^{-2x}$$

---

b) $\quad \text{I.F.} = e^{\int 2\,dx} = e^{2x}$

$$\frac{d}{dx}\left(e^{2x}u\right) = 1$$

$$e^{2x}u = x + c$$

$$u = (x + c)e^{-2x}$$

c)

$$\frac{dy}{dx} + 2y = (x+c)e^{-2x}$$

$$\frac{d}{dx}\left(e^{2x}y\right) = x + c$$

$$e^{2x}y = \frac{x^2}{2} + cx + d$$

$$y = \left(\frac{x^2}{2} + cx + d\right)e^{-2x}$$

---

5)

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = 6 + 5\sin x$$

Aux Eqn

$$m^2 - 4m + 3 = 0$$

$$(m-1)(m-3) = 0$$

C.F.

$$y = Ae^{x} + Be^{3x}$$

P.I.

$$y = \omega + \lambda\cos x + \mu\sin x$$

$$\frac{dy}{dx} = -\lambda\sin x + \mu\cos x$$

$$\frac{d^2y}{dx^2} = -\lambda\cos x - \mu\sin x$$

$$-\lambda\cos x - \mu\sin x + 4\lambda\sin x - 4\mu\cos x$$

$$+ 3\omega + 3\lambda\cos x + 3\mu\sin x = 6 + 5\sin x$$

$\Rightarrow \omega = 2$

$$-\lambda - 4\mu + 3\lambda = 0$$
$$-4\mu + 2\lambda = 0 \qquad (1)$$

$$-\mu + 4\lambda + 3\mu = 5$$
$$4\lambda + 2\mu = 5 \qquad (2)$$

$(1) \times 2$
$$4\lambda - 8\mu = 0 \qquad (3)$$

$(2) - (3)$
$$10\mu = 5$$
$$\mu = \tfrac{1}{2}$$
$$\lambda = 1$$

P.I. $\quad y = 2 + \cos x + \frac{1}{2}\sin x$

Gen Soln $\quad y = Ae^{x} + Be^{3x} + 2 + \cos x + \frac{1}{2}\sin x$

---

1) a) $\quad y = kx^2 e^{5x}$

$$\frac{dy}{dx} = 5kx^2 e^{5x} + 2kxe^{5x} = ke^{5x}(5x^2 + 2x)$$

$$\frac{d^2y}{dx^2} = ke^{5x}(10x + 2) + 5ke^{5x}(5x^2 + 2x)$$

$$\frac{d^2y}{dx^2} - 10\frac{dy}{dx} + 25y = 6e^{5x}$$

$$ke^{5x}(10x+2) + 5ke^{5x}(5x^2+2x)$$
$$-10ke^{5x}(5x^2+2x) + 25kx^2e^{5x} = 6e^{5x}$$

$$k(10x+2) + 5k(5x^2+2x)$$
$$-10k(5x^2+2x) + 25kx^2 = 6$$

coeff of x $\quad 10k + 10k - 20k = 0$ ✓

coeff of x^2 $\quad 25k - 50k + 25k = 0$ ✓

coeff of const $\quad 2k = 6$

$$\underline{k = 3} \qquad \text{P.I. } y = 3x^2e^{5x}$$

$$\frac{d^2y}{dx^2} - 10\frac{dy}{dx} + 25 = 6e^{5x}$$

Aux eqn $\quad m^2 - 10m + 25 = 0$

$(m-5)^2 = 0$

C.F. $\quad y = (A + Bx)e^{5x}$

Gen Soln $\quad y = (A + Bx)e^{5x} + 3x^2e^{5x}$