

11 A cubic curve has equation $y = x^3 - 3x^2 + 1$.

(i) Use calculus to find the coordinates of the turning points on this curve. Determine the nature of these turning points. [5]

(ii) Show that the tangent to the curve at the point where $x = -1$ has gradient 9.

Find the coordinates of the other point, P, on the curve at which the tangent has gradient 9 and find the equation of the normal to the curve at P.

Show that the area of the triangle bounded by the normal at P and the x- and y-axes is 8 square units. [8]

$$\text{i) } \frac{dy}{dx} = 3x^2 - 6x$$

$$\text{At st. pt. } \frac{dy}{dx} = 0 \Rightarrow 3x^2 - 6x = 0 \\ 3x(x - 2) = 0 \\ x = 0 \text{ or } x = 2$$

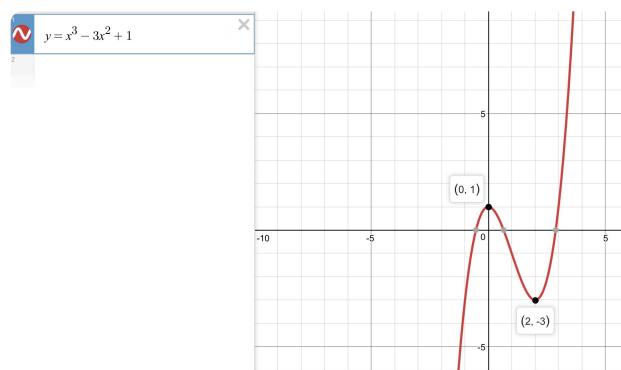
$$\begin{cases} x = 0 \\ y = 1 \end{cases} \quad \begin{cases} x = 2 \\ y = 2^3 - 3(2)^2 + 1 \\ y = 8 - 12 + 1 \\ y = -3 \end{cases}$$

$$\frac{d^2y}{dx^2} = 6x - 6 \quad x = 0, \frac{d^2y}{dx^2} = -6 < 0 \text{ max}$$

$$x = 2, \frac{d^2y}{dx^2} = 12 - 6 = 6 > 0 \text{ min}$$

max at $(0, 1)$

min at $(2, -3)$



$$\text{ii) } y = x^3 - 3x^2 + 1$$

$$\frac{dy}{dx} = 3x^2 - 6x$$

$$\text{when } x = -1, \quad \begin{aligned}\frac{dy}{dx} &= 3(-1)^2 - 6(-1) \\ &= 3 + 6 \\ &= 9\end{aligned}$$

\therefore gradient of tangent = 9 when $x = -1$

$$\begin{aligned}\frac{dy}{dx} &= 3x^2 - 6x = 9 \\ 3x^2 - 6x - 9 &= 0 \\ x^2 - 2x - 3 &= 0 \\ (x+1)(x-3) &= 0 \\ \Rightarrow x &= -1 \text{ or } x = 3\end{aligned}$$

$\therefore P$ has x -coord = 3

$$x = 3, \quad y = 3^3 - 3(3)^2 + 1 = 1$$

$$\therefore P(3, 1)$$

Normal at P has gradient = $-\frac{1}{9}$

$$y - y_1 = m(x - x_1)$$

$$y - 1 = -\frac{1}{9}(x - 3)$$

$$y - 1 = -\frac{1}{9}x + \frac{1}{3}$$

$$y = -\frac{1}{9}x + \frac{4}{3}$$

$$\text{When } x = 0 \quad y = \frac{4}{3}$$

$$\text{when } y = 0 \quad -\frac{1}{3}x + \frac{4}{3} = 0$$

$$\frac{4}{3} = \frac{1}{3}x$$

$$12 = x$$

Cuts x -axis at $(12, 0)$

Cuts y -axis at $(0, \frac{4}{3})$

$$\text{Area of triangle} = \frac{1}{2} \text{base} \times \text{height}$$

$$= \frac{1}{2} \times 12 \times \frac{4}{3}$$

$$= 8 \text{ units}^2$$

5

10 A curve has equation $y = x^3 - 6x^2 + 12$.

(i) Use calculus to find the coordinates of the turning points of this curve. Determine also the nature of these turning points. [7]

(ii) Find, in the form $y = mx + c$, the equation of the normal to the curve at the point $(2, -4)$. [4]

i) $y = x^3 - 6x^2 + 12$

$$\frac{dy}{dx} = 3x^2 - 12x$$

$$\text{At a E.p. } \frac{dy}{dx} = 0 \Rightarrow 3x^2 - 12x = 0$$

$$3x(x - 4) = 0$$

$$x = 0 \text{ or } x = 4$$

$$\begin{cases} x = 0 \\ y = 12 \end{cases} \quad \begin{cases} x = 4 \\ y = 4^3 - 6(4)^2 + 12 \end{cases}$$

$$\begin{cases} x = 4 \\ y = 64 - 96 + 12 \\ y = -20 \end{cases}$$

$$\frac{d^2y}{dx^2} = 6x - 12$$

$x=0, \frac{d^2y}{dx^2} = -12 < 0 \therefore \text{max}$

$x=4, \frac{d^2y}{dx^2} = +12 > 0 \therefore \text{min}$

maximum at $(0, 12)$

minimum at $(4, -20)$

when $x = 2, \frac{dy}{dx} = 3(2)^2 - 12(2) = -12$

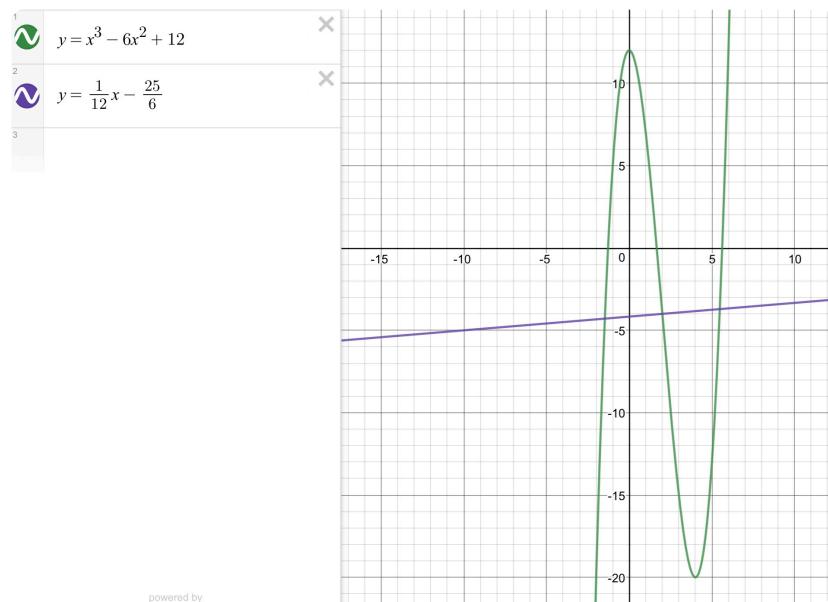
\therefore gradient of normal $= +\frac{1}{12}$ Point is $(2, -4)$

Normal $y - y_1 = m(x - x_1)$

$$y - -4 = \frac{1}{12}(x - 2)$$

$$y + 4 = \frac{1}{12}x - \frac{1}{6}$$

$$y = \frac{1}{12}x - \frac{25}{6}$$



10 The equation of a curve is $y = 7 + 6x - x^2$.

(i) Use calculus to find the coordinates of the turning point on this curve.

Find also the coordinates of the points of intersection of this curve with the axes, and sketch the curve. [8]

$$y = 7 + 6x - x^2$$

Cuts y -axis at $(0, 7)$

$$\frac{dy}{dx} = 6 - 2x$$

Cuts x -axis when

At t.p. $\frac{dy}{dx} = 0$

$$7 + 6x - x^2 = 0$$

$$\Rightarrow 6 - 2x = 0$$

$$x^2 - 6x - 7 = 0$$

$$6 = 2x$$

$$(x+1)(x-7) = 0$$

$$3 = x$$

$$x = -1 \text{ or } x = 7$$

$$y = 7 + 6(3) - 3^2$$

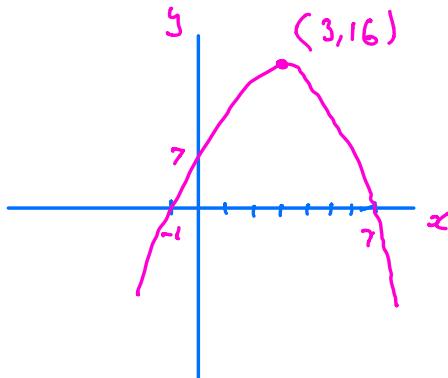
$$(-1, 0) \text{ and } (7, 0)$$

$$y = 7 + 18 - 9$$

$$y = 16$$

Turning point at

$$(3, 16)$$



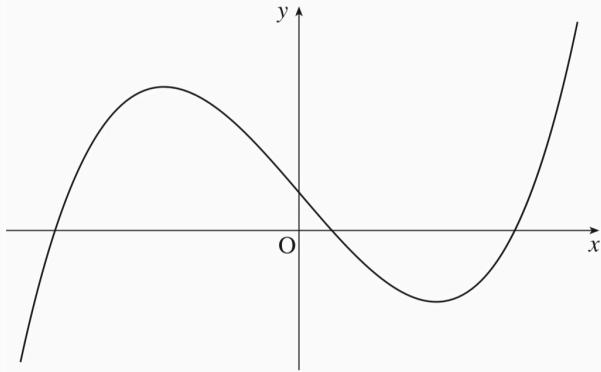


Fig. 11

The equation of the curve shown in Fig. 11 is $y = x^3 - 6x + 2$.

(i) Find $\frac{dy}{dx}$. [2]

(ii) Find, in exact form, the range of values of x for which $x^3 - 6x + 2$ is a decreasing function. [3]

(iii) Find the equation of the tangent to the curve at the point $(-1, 7)$.

Find also the coordinates of the point where this tangent crosses the curve again. [6]

i) $\frac{dy}{dx} = 3x^2 - 6$

ii) At st.pt. $\frac{dy}{dx} = 0 \Rightarrow 3x^2 - 6 = 0$
 $x^2 - 2 = 0$
 $x = \pm\sqrt{2}$

Decreasing Function for $-\sqrt{2} < x < \sqrt{2}$

iii) When $x = -1$, $\frac{dy}{dx} = 3(-1)^2 - 6 = -3$

Point $(-1, 7)$

$$\begin{aligned}y - y_1 &= m(x - x_1) \\y - 7 &= -3(x - -1)\end{aligned}$$

$$\begin{aligned}
 y - 7 &= -3(x+1) \\
 y - 7 &= -3x - 3 \\
 y &= -3x + 4
 \end{aligned}$$

$$\begin{cases} y = x^3 - 6x + 2 \\ y = -3x + 4 \end{cases}$$

$$x^3 - 6x + 2 = -3x + 4$$

$$x^3 - 3x - 2 = 0$$

$$(x+1)(x+1)(x-2) = 0$$

$$\Rightarrow x = 2$$

$$x = 2, \quad y = 2^3 - 6(2) + 2$$

$$y = 8 - 12 + 2$$

$$y = -2$$

Meets curve at $(2, -2)$
