Section B (36 marks)

7 Data suggest that the number of cases of infection from a particular disease tends to oscillate between two values over a period of approximately 6 months.

(a) Suppose that the number of cases, P thousand, after time t months is modelled by the equation $P = \dfrac{2}{2 - \sin t}$. Thus, when $t = 0, P = 1$.

(i) By considering the greatest and least values of $\sin t$, write down the greatest and least values of P predicted by this model. [2]

(ii) Verify that P satisfies the differential equation $\dfrac{\mathrm{d}P}{\mathrm{d}t} = \frac{1}{2} P^2 \cos t$. [5]

(b) An alternative model is proposed, with differential equation

$$\frac{\mathrm{d}P}{\mathrm{d}t} = \frac{1}{2}(2P^2 - P)\cos t. \qquad (*)$$

As before, $P = 1$ when $t = 0$.

(i) Express $\dfrac{1}{P(2P-1)}$ in partial fractions. [4]

(ii) Solve the differential equation (*) to show that

$$\ln\left(\frac{2P-1}{P}\right) = \frac{1}{2}\sin t. \qquad [5]$$

This equation can be rearranged to give $P = \dfrac{1}{2 - \mathrm{e}^{\frac{1}{2}\sin t}}$.

(iii) Find the greatest and least values of P predicted by this model. [4]

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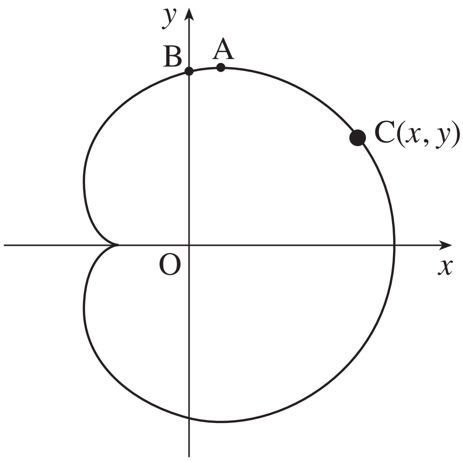

Fig. 8

In a theme park ride, a capsule C moves in a vertical plane (see Fig. 8). With respect to the axes shown, the path of C is modelled by the parametric equations

$$x = 10\cos\theta + 5\cos 2\theta, \quad y = 10\sin\theta + 5\sin 2\theta, \qquad (0 \leqslant \theta < 2\pi),$$

where x and y are in metres.

(i) Show that $\dfrac{dy}{dx} = -\dfrac{\cos\theta + \cos 2\theta}{\sin\theta + \sin 2\theta}$.

Verify that $\dfrac{dy}{dx} = 0$ when $\theta = \frac{1}{3}\pi$. Hence find the exact coordinates of the highest point A on the path of C. [6]

(ii) Express $x^2 + y^2$ in terms of θ. Hence show that

$$x^2 + y^2 = 125 + 100\cos\theta. \qquad [4]$$

(iii) Using this result, or otherwise, find the greatest and least distances of C from O. [2]

You are given that, at the point B on the path vertically above O,

$$2\cos^2\theta + 2\cos\theta - 1 = 0.$$

(iv) Using this result, and the result in part (**ii**), find the distance OB. Give your answer to 3 significant figures. [4]