Section B (36 marks)

6 In Fig. 6, OAB is a thin bent rod, with OA = a metres, AB = b metres and angle OAB = $120°$. The bent rod lies in a vertical plane. OA makes an angle θ above the horizontal. The vertical height BD of B above O is h metres. The horizontal through A meets BD at C and the vertical through A meets OD at E.

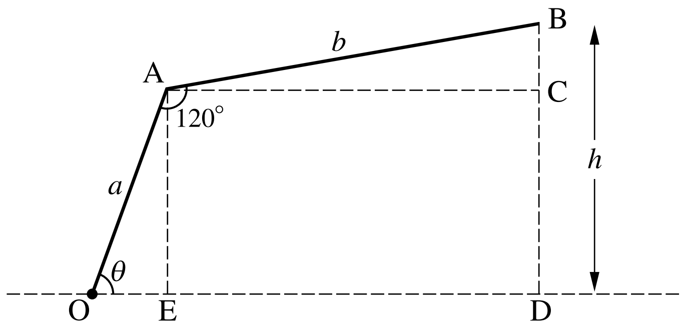

Fig. 6

(i) Find angle BAC in terms of θ. Hence show that

$$h = a\sin\theta + b\sin(\theta - 60°).$$ **[3]**

(ii) Hence show that $h = (a + \frac{1}{2}b)\sin\theta - \frac{\sqrt{3}}{2}b\cos\theta$. **[3]**

The rod now rotates about O, so that θ varies. You may assume that the formulae for h in parts **(i)** and **(ii)** remain valid.

(iii) Show that OB is horizontal when $\tan\theta = \frac{\sqrt{3}b}{2a + b}$. **[3]**

In the case when $a = 1$ and $b = 2$, $h = 2\sin\theta - \sqrt{3}\cos\theta$.

(iv) Express $2\sin\theta - \sqrt{3}\cos\theta$ in the form $R\sin(\theta - \alpha)$. Hence, for this case, write down the maximum value of h and the corresponding value of θ. **[7]**

[Question 7 is printed overleaf.]

7 Fig. 7 illustrates the growth of a population with time. The proportion of the ultimate (long term) population is denoted by x, and the time in years by t. When $t = 0$, $x = 0.5$, and as t increases, x approaches 1.

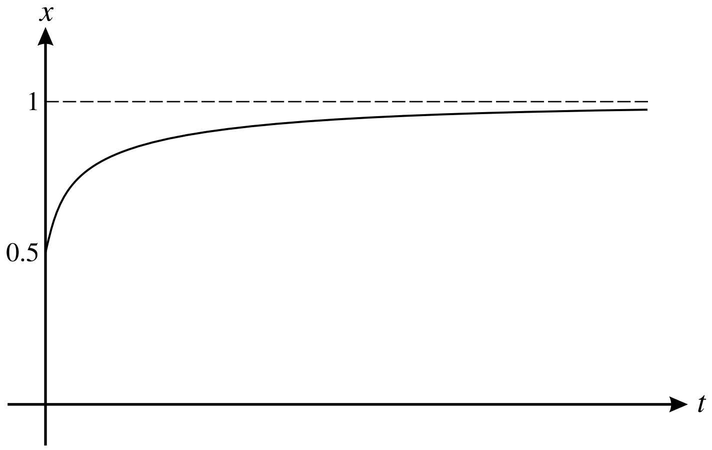

Fig. 7

One model for this situation is given by the differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x(1 - x).$$

(i) Verify that $x = \dfrac{1}{1 + \mathrm{e}^{-t}}$ satisfies this differential equation, including the initial condition. **[6]**

(ii) Find how long it will take, according to this model, for the population to reach three-quarters of its ultimate value. **[3]**

An alternative model for this situation is given by the differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x^2(1 - x),$$

with $x = 0.5$ when $t = 0$ as before.

(iii) Find constants A, B and C such that $\dfrac{1}{x^2(1-x)} = \dfrac{A}{x^2} + \dfrac{B}{x} + \dfrac{C}{1-x}$. **[4]**

(iv) Hence show that $t = 2 + \ln\left(\dfrac{x}{1-x}\right) - \dfrac{1}{x}$. **[5]**

(v) Find how long it will take, according to this model, for the population to reach three-quarters of its ultimate value. **[2]**