Section B (36 marks)

7 In a game of rugby, a kick is to be taken from a point P (see Fig. 7). P is a perpendicular distance y metres from the line TOA. Other distances and angles are as shown.

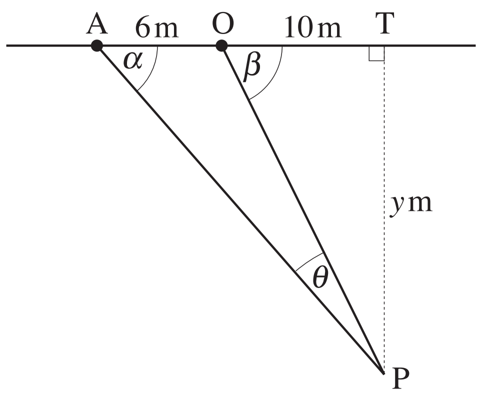

Fig. 7

(i) Show that $\theta = \beta - \alpha$, and hence that $\tan\theta = \dfrac{6y}{160 + y^2}$.

Calculate the angle θ when $y = 6$. [8]

(ii) By differentiating implicitly, show that $\dfrac{d\theta}{dy} = \dfrac{6(160 - y^2)}{(160 + y^2)^2}\cos^2\theta$. [5]

(iii) Use this result to find the value of y that maximises the angle θ. Calculate this maximum value of θ. [You need not verify that this value is indeed a maximum.] [4]

[Question 8 is printed overleaf.]

8 Some years ago an island was populated by red squirrels and there were no grey squirrels. Then grey squirrels were introduced.

The population x, in thousands, of red squirrels is modelled by the equation

$$x = \frac{a}{1 + kt},$$

where t is the time in years, and a and k are constants. When $t = 0, x = 2.5$.

(i) Show that $\frac{dx}{dt} = -\frac{kx^2}{a}$. [3]

(ii) Given that the initial population of 2.5 thousand red squirrels reduces to 1.6 thousand after one year, calculate a and k. [3]

(iii) What is the long-term population of red squirrels predicted by this model? [1]

The population y, in thousands, of grey squirrels is modelled by the differential equation

$$\frac{dy}{dt} = 2y - y^2.$$

When $t = 0, y = 1$.

(iv) Express $\frac{1}{2y - y^2}$ in partial fractions. [4]

(v) Hence show by integration that $\ln\left(\frac{y}{2 - y}\right) = 2t$.

Show that $y = \frac{2}{1 + e^{-2t}}$. [7]

(vi) What is the long-term population of grey squirrels predicted by this model? [1]