

E is  $(2a\pi, 0)$

$$\therefore OE = 2a\pi - 0 = 2a\pi$$

B)

$$\text{Max } y = a(1 - \cos\theta)$$

$$\text{when } \cos\theta = -1$$

$$\text{Max height} = a(1 - -1) = 2a$$

ii)

$$\frac{dy}{d\theta} = +a \sin\theta$$

$$\frac{dx}{d\theta} = a(1 - \cos\theta)$$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

$$= \frac{a \sin\theta}{a(1 - \cos\theta)} = \frac{\sin\theta}{1 - \cos\theta}$$

6)

$$x = a(\theta - \sin\theta), y = a(1 - \cos\theta)$$

i)  
A)

Find OE

At E x has max value

$$y = 0$$

$$y = 0 \Rightarrow 1 - \cos\theta = 0$$

$$\Rightarrow \cos\theta = 1$$

$$\Rightarrow \theta = 0 \text{ or } 2\pi$$

O is point  $(0, 0)$

E is point  $(a(2\pi - \sin 2\pi), 0)$

iii)

At B, AB is a tgt

$$\text{Grad of AB} = \tan^{-1} 30^\circ = \frac{1}{\sqrt{3}}$$

$$\therefore \text{gradient of curve at B} = \frac{1}{\sqrt{3}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{3}}$$

$$\Rightarrow \frac{\sin\theta}{1 - \cos\theta} = \frac{1}{\sqrt{3}}$$

$$\Rightarrow \sin\theta = \frac{1}{\sqrt{3}}(1 - \cos\theta)$$

6(iii)  
cont)

$$\sin \frac{2\pi}{3} = \frac{\sqrt{3}}{2}$$

$$\begin{aligned}\frac{1}{\sqrt{3}}(1 - \cos \frac{2\pi}{3}) &= \frac{1}{\sqrt{3}}\left(1 - -\frac{1}{2}\right) \\ &= \frac{1}{\sqrt{3}} \times \frac{3}{2} = \frac{\sqrt{3}}{2}\end{aligned}$$

$\therefore \theta = \frac{2\pi}{3}$  is a solution of

$$\sin \theta = \frac{1}{\sqrt{3}}(1 - \cos \theta)$$


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BF is y coord of B

$$y = a(1 - \cos \theta)$$

at B,  $\theta = \frac{2\pi}{3}$  so

$$\begin{aligned}y &= a\left(1 - \cos \frac{2\pi}{3}\right) \\ &= a\left(1 - -\frac{1}{2}\right) = \frac{3a}{2}\end{aligned}$$

$$\Rightarrow BF = \frac{3a}{2}$$

OF is same as x coord of B

$$x = a(\theta - \sin \theta)$$

$$x = a\left(\frac{2\pi}{3} - \sin \frac{2\pi}{3}\right)$$

$$x = a\left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right)$$

$$\therefore OF = a\left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right)$$


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(iv) Due to symmetry

$$BC = OE - 2 \times OF$$

$$= 2a\pi - 2\left(a\left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right)\right)$$

$$= 2a\pi - \frac{4a\pi}{3} + \sqrt{3}a$$

$$= \frac{2a\pi}{3} + \sqrt{3}a$$

$$BC = a\left(\frac{2\pi}{3} + \sqrt{3}\right)$$


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$$\tan 30^\circ = \frac{1}{\sqrt{3}} = \frac{BF}{AF}$$

$$\Rightarrow AF = \sqrt{3} BF$$

$$AF = \sqrt{3} \times \frac{3a}{2}$$

$$AF = \frac{3\sqrt{3}a}{2}$$


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Due to symmetry

$$AD = 2AF + BC$$

$$= 2\left(\frac{3\sqrt{3}a}{2}\right) + a\left(\frac{2\pi}{3} + \sqrt{3}\right)$$

$$\therefore AD = 3\sqrt{3}a + a\left(\frac{2\pi}{3} + \sqrt{3}\right)$$

$$AD = a\left(4\sqrt{3} + \frac{2\pi}{3}\right)$$

$$a = \frac{20}{\left(4\sqrt{3} + \frac{2\pi}{3}\right)} = 2.22 \text{ m to 3 s.f.}$$

7i)  $A(0,0,6) \quad E(15, -20, 6)$

$$|AE| = \sqrt{15^2 + (-20)^2 + (6-6)^2}$$

$$|AE| = 25 \text{ m}$$

$$-3(0) + 4(0) + 5(6) = 30 \quad \checkmark$$

$$-3(-1) + 4(-7) + 5(11)$$

$$= 3 - 28 + 55 = 30 \quad \checkmark$$

$$-3(-8) + 4(-6) + 5(6)$$

$$= 24 - 24 + 30 = 30 \quad \checkmark$$

7ii) Direction BD same as direction AE

$$\vec{AE} = \begin{pmatrix} 15 \\ -20 \\ 0 \end{pmatrix} = 5 \begin{pmatrix} 3 \\ -4 \\ 0 \end{pmatrix}$$

Vector eqn of BD

$$\underline{r} = \begin{pmatrix} -1 \\ -7 \\ 11 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -4 \\ 0 \end{pmatrix}$$

$$|BD| = 15 \Rightarrow \left| \lambda \begin{pmatrix} 3 \\ -4 \\ 0 \end{pmatrix} \right| = 15$$

$$\lambda \sqrt{3^2 + (-4)^2 + 0^2} = 15$$

$$5\lambda = 15$$

$$\lambda = 3$$

D is given by

$$\underline{r} = \begin{pmatrix} -1 \\ -7 \\ 11 \end{pmatrix} + 3 \begin{pmatrix} 3 \\ -4 \\ 0 \end{pmatrix} = \begin{pmatrix} 8 \\ -19 \\ 11 \end{pmatrix}$$

$$D(8, -19, 11)$$

$$-3(0) + 4(0) + 5(6) = 30 \quad \checkmark$$

$$-3(-1) + 4(-7) + 5(11)$$

$$= 3 - 28 + 55 = 30 \quad \checkmark$$

$$-3(-8) + 4(-6) + 5(6)$$

$$= 24 - 24 + 30 = 30 \quad \checkmark$$

A, B, C on plane  $-3x + 4y + 5z = 30$

A Normal is  $\begin{pmatrix} -3 \\ 4 \\ 5 \end{pmatrix}$

iv)  $\vec{BD} = 3 \begin{pmatrix} 3 \\ -4 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} 4 \\ 3 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -4 \\ 0 \end{pmatrix} = 12 - 12 + 0 = 0$$

$\therefore \perp$

$$\vec{AB} = \begin{pmatrix} -1 \\ -7 \\ 5 \end{pmatrix}$$

$$\begin{pmatrix} 4 \\ 3 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ -7 \\ 5 \end{pmatrix} = -4 - 21 + 25 = 0$$

$\therefore \perp$

Since  $\begin{pmatrix} 4 \\ 3 \\ 5 \end{pmatrix}$  is  $\perp$  to two non-parallel vectors within

plane ABDE, it is normal to the plane.

Plane of form  $4x + 3y + 5z = d$

A on plane so  $4(0) + 3(0) + 5 \times 6 = d$

$$\Rightarrow d = 30$$

Plane is  $4x + 3y + 5z = 30$

7iii)

$$A(0,0,6)$$

$$B(-1,-7,11)$$

$$C(-8,-6,6)$$

7v) Angle between planes is  
same as angle between normals  
to the planes

$$\cos \theta = \frac{\begin{pmatrix} -3 \\ 4 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 3 \\ 5 \end{pmatrix}}{\left| \begin{pmatrix} -3 \\ 4 \\ 5 \end{pmatrix} \right| \left| \begin{pmatrix} 4 \\ 3 \\ 5 \end{pmatrix} \right|}$$

$$\cos \theta = \frac{-12 + 12 + 25}{\sqrt{(-3)^2 + 4^2 + 5^2} \sqrt{4^2 + 3^2 + 5^2}}$$

$$\cos \theta = \frac{25}{\sqrt{50} \sqrt{50}}$$

$$\cos \theta = \frac{25}{50} = \frac{1}{2}$$

$$\Rightarrow \theta = 60^\circ$$

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