

$$\begin{aligned}6(i)) \angle BAC &= 120 - (90 - \alpha) - 90 \\&= \alpha - 60^\circ\end{aligned}$$

$$h = BC + CD = BC + AE$$

$$h = b \sin(\angle BAC) + a \sin \alpha$$

$$h = b \sin(\alpha - 60^\circ) + a \sin \alpha$$

$$6(ii)) h = a \sin \alpha + b \left( \sin \alpha \cos 60^\circ - \cos \alpha \sin 60^\circ \right)$$

$$h = a \sin \alpha + \frac{b}{2} \sin \alpha - \frac{\sqrt{3}}{2} b \cos \alpha$$

$$h = \left( a + \frac{b}{2} \right) \sin \alpha - \frac{\sqrt{3}}{2} b \cos \alpha$$

6(iii)) When OB is horizontal  $h = 0$

$$\Rightarrow \left( a + \frac{b}{2} \right) \sin \alpha - \frac{\sqrt{3}}{2} b \cos \alpha = 0$$

$$\left( a + \frac{b}{2} \right) \sin \alpha = \frac{\sqrt{3}}{2} b \cos \alpha$$

$$\Rightarrow \tan \alpha = \frac{\sin \alpha}{\cos \alpha} = \frac{\sqrt{3}b/2}{a+b/2}$$

$$\tan \alpha = \frac{\sqrt{3}b}{2a+b}$$

$$6(iv)) 2 \sin \alpha - \sqrt{3} \cos \alpha$$

$$\begin{aligned}\text{Diagram: } &\text{A right-angled triangle with hypotenuse } \sqrt{7}, \text{ vertical leg } \sqrt{3}, \text{ and horizontal leg } 2. \\&\angle \alpha \text{ is at the top-left vertex.} \\&\angle \alpha = \tan^{-1} \left( \frac{\sqrt{3}}{2} \right) = 40.9^\circ \\&2 \sin \alpha - \sqrt{3} \cos \alpha = \sqrt{7} \left( \frac{2 \sin \alpha - \sqrt{3} \cos \alpha}{\sqrt{7}} \right) \\&= \sqrt{7} \sin(\alpha - \alpha)\end{aligned}$$

$$\text{where } \alpha = \tan^{-1} \left( \frac{\sqrt{3}}{2} \right) = 40.9^\circ$$

$$= \sqrt{7} \sin(\alpha - 40.9^\circ)$$

$$h = \sqrt{7} \sin(\alpha - 40.9^\circ)$$

Max value of  $h = \sqrt{7}$  m

Occurs when  $\alpha - 40.9^\circ = 90^\circ$

$$\Rightarrow \alpha = 130.9^\circ$$

$$7) \frac{dx}{dt} = x(1-x)$$

$$\text{i) Consider } x = \frac{1}{1+e^{-t}}$$

$$\text{when } t=0, x = \frac{1}{1+1} = 0.5$$

$\therefore$  initial condition satisfied

$$\text{if } x = \frac{1}{1+e^{-t}} = (1+e^{-t})^{-1}$$

$$\begin{aligned}\frac{dx}{dt} &= -1(1+e^{-t})^{-2} \times (-e^{-t}) \\&= \frac{e^{-t}}{(1+e^{-t})^2} \quad (\star)\end{aligned}$$

Also

$$x(1-x) = x - x^2$$

$$= \frac{1}{1+e^{-t}} - \frac{1}{(1+e^{-t})^2}$$

$$= \frac{1+e^{-t}-1}{(1+e^{-t})^2} = \frac{e^{-t}}{(1+e^{-t})^2}$$

This is the same as  $(\star)$

$\therefore x = \frac{1}{1+e^{-t}}$  satisfies  
the diff. eqn.

7ii) Require  $x = \frac{3}{4}$

$$\frac{3}{4} = \frac{1}{1+e^{-t}}$$

$$3(1+e^{-t}) = 4$$

$$1+e^{-t} = \frac{4}{3}$$

$$e^{-t} = \frac{1}{3}$$

$$-t = \ln \frac{1}{3}$$

$$t = \ln 3$$

$$t = 1.10 \text{ years}$$

7iii)

$$\frac{1}{x^2(1-x)} = \frac{A}{x^2} + \frac{B}{x} + \frac{C}{1-x}$$

$$1 = A(1-x) + Bx(1-x) + Cx^2$$

$$x=1$$

$$\Rightarrow 1 = C$$

$$x=0$$

$$\Rightarrow 1 = A$$

coeff of  $x^2$ 

$$0 = -B + C$$

$$\Rightarrow B=C \quad \therefore B=1$$

$$\frac{1}{x^2(1-x)} = \frac{1}{x^2} + \frac{1}{x} + \frac{1}{1-x}$$

7iv)

$$\frac{dx}{dt} = x^2(1-x)$$

$$\Rightarrow \int \frac{1}{x^2(1-x)} dx = \int 1 dt$$

$$\int \left( \frac{1}{x^2} + \frac{1}{x} + \frac{1}{1-x} \right) dx = \int 1 dt$$

$$-\frac{1}{x} + \ln x - \ln(1-x) + C = t$$

$$t=0, x=0.5 \text{ given}$$

$$-2 + \ln \frac{1}{2} - \ln \frac{1}{2} + C = 0$$

$$\Rightarrow C = 2$$

$$\therefore t = -\frac{1}{x} + \ln \left( \frac{x}{1-x} \right) + 2$$

7v) Require  $x = \frac{3}{4}$

$$t = -\frac{4}{3} + \ln \left( \frac{\frac{3}{4}}{1-\frac{3}{4}} \right) + 2$$

$$t = \frac{2}{3} + \ln(3)$$

$$t = 1.77 \text{ years}$$