6 A curve has cartesian equation $y^2 - x^2 = 4$.

(i) Verify that

$$x = t - \frac{1}{t}, \quad y = t + \frac{1}{t},$$

are parametric equations of the curve. [2]

(ii) Show that $\frac{dy}{dx} = \frac{(t-1)(t+1)}{t^2+1}$. Hence find the coordinates of the stationary points of the curve. [6]

Jan 2006

2 A curve is defined parametrically by the equations

$$x = t - \ln t, \qquad y = t + \ln t \qquad (t > 0).$$

Find the gradient of the curve at the point where $\quad = 2$. [5]

Jun 2006

6 Fig. 6 shows the arch ABCD of a bridge.

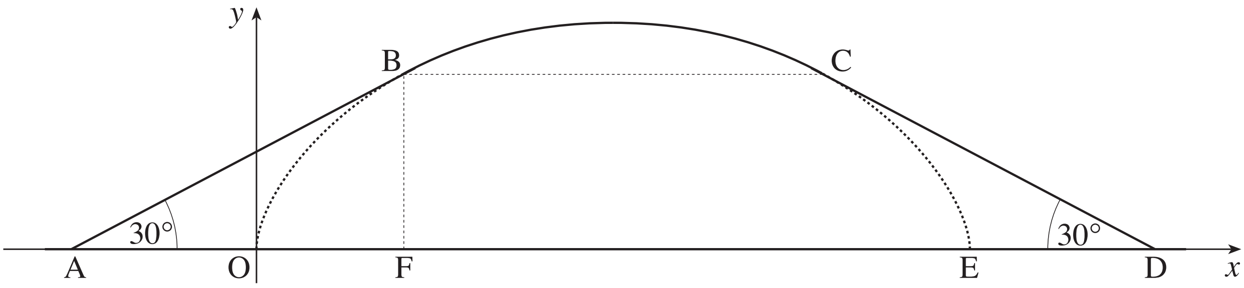

Fig. 6

The section from B to C is part of the curve OBCE with parametric equations

$$x = a(\theta - \sin\theta),\ y = a(1 - \cos\theta) \text{ for } 0 \leqslant \theta \leqslant 2\pi,$$

where a is a constant.

(i) Find, in terms of a,

(A) the length of the straight line OE,

(B) the maximum height of the arch. [4]

(ii) Find $\frac{dy}{dx}$ in terms of θ. [3]

The straight line sections AB and CD are inclined at 30° to the horizontal, and are tangents to the curve at B and C respectively. BC is parallel to the x-axis. BF is parallel to the y-axis.

(iii) Show that at the point B the parameter θ satisfies the equation

$$\sin\theta = \frac{1}{\sqrt{3}}(1 - \cos\theta).$$

Verify that $\theta = \frac{2}{3}\pi$ is a solution of this equation.

Hence show that BF $= \frac{3}{2}a$, and find OF in terms of a, giving your answer exactly. [6]

(iv) Find BC and AF in terms of a.

Given that the straight line distance AD is 20 metres, calculate the value of a. [5]

7 Fig. 7 shows the curve with parametric equations

$$x = \cos\theta,\ y = \sin\theta - \tfrac{1}{8}\sin 2\theta,\ 0 \leqslant \theta < 2\pi.$$

The curve crosses the x-axis at points $A(1, 0)$ and $B(-1, 0)$, and the positive y-axis at C. D is the maximum point of the curve, and E is the minimum point.

The solid of revolution formed when this curve is rotated through 360° about the x-axis is used to model the shape of an egg.

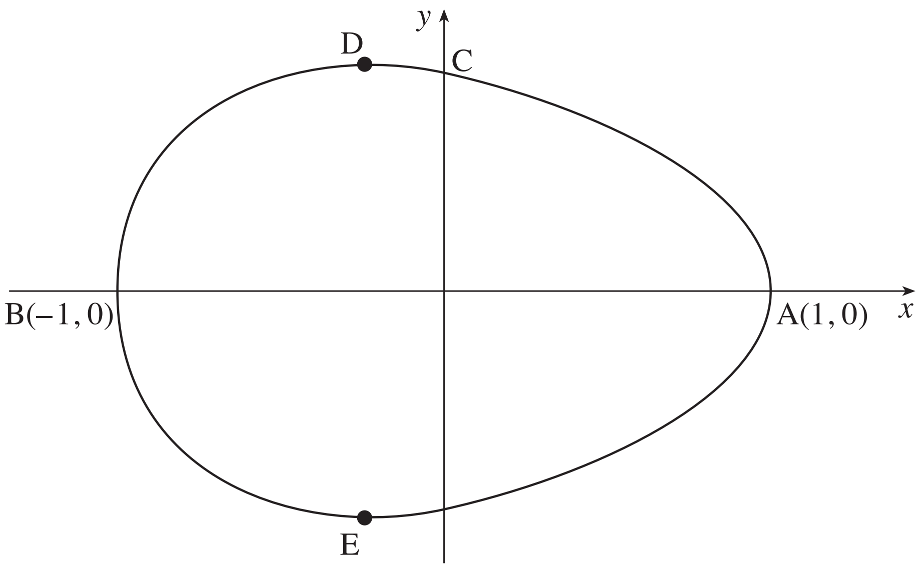

Fig. 7

(i) Show that, at the point A, $\theta = 0$. Write down the value of θ at the point B, and find the coordinates of C. [4]

(ii) Find $\dfrac{dy}{dx}$ in terms of θ.

Hence show that, at the point D,

$$2\cos^2\theta - 4\cos\theta - 1 = 0.$$ [5]

(iii) Solve this equation, and hence find the y-coordinate of D, giving your answer correct to 2 decimal places. [5]

The cartesian equation of the curve (for $0 \leqslant \theta \leqslant \pi$) is

$$y = \tfrac{1}{4}(4 - x)\sqrt{1 - x^2}.$$

(iv) Show that the volume of the solid of revolution of this curve about the x-axis is given by

$$\tfrac{1}{16}\pi\int_{-1}^{1}\left(16 - 8x - 15x^2 + 8x^3 - x^4\right)dx.$$

Evaluate this integral. [6]

8

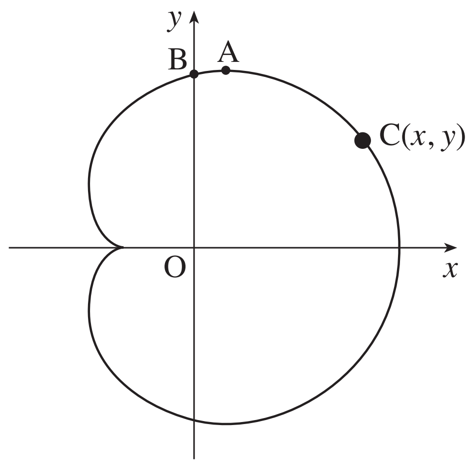

Fig. 8

In a theme park ride, a capsule C moves in a vertical plane (see Fig. 8). With respect to the axes shown, the path of C is modelled by the parametric equations

$$x = 10\cos\theta + 5\cos 2\theta, \quad y = 10\sin\theta + 5\sin 2\theta, \qquad (0 \leqslant \theta < 2\pi),$$

where x and y are in metres.

(i) Show that $\dfrac{dy}{dx} = -\dfrac{\cos\theta + \cos 2\theta}{\sin\theta + \sin 2\theta}$.

Verify that $\dfrac{dy}{dx} = 0$ when $\theta = \frac{1}{3}\pi$. Hence find the exact coordinates of the highest point A on the path of C. [6]

(ii) Express $x^2 + y^2$ in terms of θ. Hence show that

$$x^2 + y^2 = 125 + 100\cos\theta. \qquad [4]$$

(iii) Using this result, or otherwise, find the greatest and least distances of C from O. [2]

You are given that, at the point B on the path vertically above O,

$$2\cos^2\theta + 2\cos\theta - 1 = 0.$$

(iv) Using this result, and the result in part **(ii)**, find the distance OB. Give your answer to 3 significant figures. [4]

8 A curve has equation

$$x^2 + 4y^2 = k^2,$$

where k is a positive constant.

(i) Verify that

$$x = k\cos\theta, \qquad y = \tfrac{1}{2}k\sin\theta,$$

are parametric equations for the curve. [3]

(ii) Hence or otherwise show that $\frac{dy}{dx} = -\frac{x}{4y}$. [3]

(iii) Fig. 8 illustrates the curve for a particular value of k. Write down this value of k. [1]

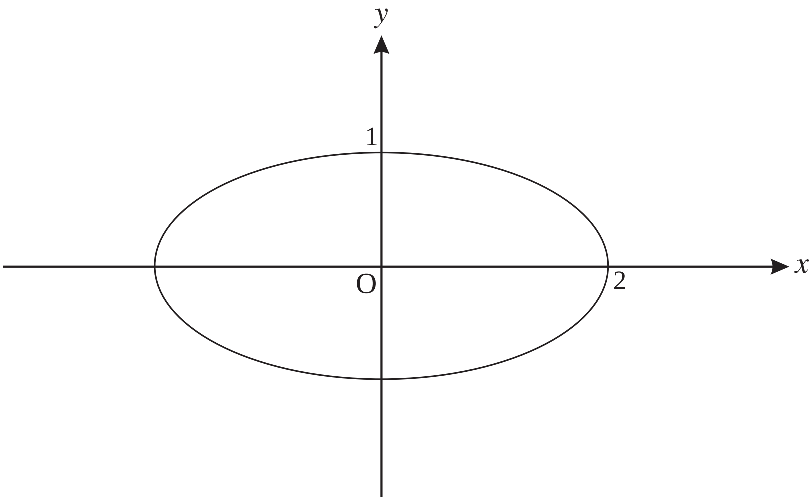

Fig. 8

(iv) Copy Fig. 8 and on the same axes sketch the curves for $k = 1$, $k = 3$ and $k = 4$. [3]

On a map, the curves represent the contours of a mountain. A stream flows down the mountain. Its path on the map is always at right angles to the contour it is crossing.

(v) Explain why the path of the stream is modelled by the differential equation

$$\frac{dy}{dx} = \frac{4y}{x}.$$ [2]

(vi) Solve this differential equation.

Given that the path of the stream passes through the point $(2, 1)$, show that its equation is $y = \frac{x^4}{16}$. [6]

5 A curve has parametric equations $x = 1 + u^2$, $y = 2u^3$.

(i) Find $\dfrac{dy}{dx}$ in terms of u. [3]

(ii) Hence find the gradient of the curve at the point with coordinates $(5, 16)$. [2]

8 Fig. 8 illustrates a hot air balloon on its side. The balloon is modelled by the volume of revolution about the x-axis of the curve with parametric equations

$$x = 2 + 2\sin\theta, \quad y = 2\cos\theta + \sin 2\theta, \quad (0 \leqslant \theta \leqslant 2\pi).$$

The curve crosses the x-axis at the point A (4, 0). B and C are maximum and minimum points on the curve. Units on the axes are metres.

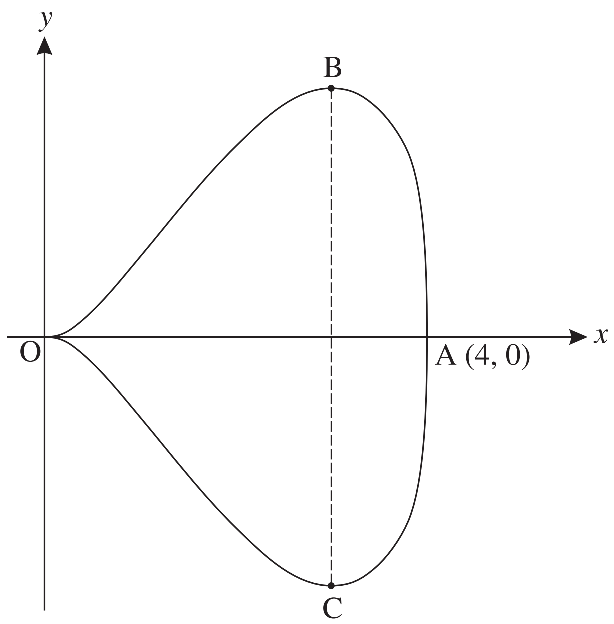

Fig. 8

(i) Find $\frac{dy}{dx}$ in terms of θ. **[4]**

(ii) Verify that $\frac{dy}{dx} = 0$ when $\theta = \frac{1}{6}\pi$, and find the exact coordinates of B.

Hence find the maximum width BC of the balloon. **[5]**

(iii) (A) Show that $y = x\cos\theta$.

(B) Find $\sin\theta$ in terms of x and show that $\cos^2\theta = x - \frac{1}{4}x^2$.

(C) Hence show that the cartesian equation of the curve is $y^2 = x^3 - \frac{1}{4}x^4$. **[7]**

(iv) Find the volume of the balloon. **[3]**

Jun 2009

5 A curve has parametric equations

$$x = at^3, \quad y = \frac{a}{1+t^2},$$

where a is a constant.

Show that $\dfrac{dy}{dx} = \dfrac{-2}{3t(1+t^2)^2}$.

Hence find the gradient of the curve at the point $(a, \frac{1}{2}a)$. **[7]**