7 In a chemical process, the mass M grams of a chemical at time t minutes is modelled by the differential equation

$$\frac{\mathrm{d}M}{\mathrm{d}t} = \frac{M}{t(1+t^2)}.$$

(i) Find $\int \frac{t}{1+t^2}\,\mathrm{d}t$. [3]

(ii) Find constants A, B and C such that

$$\frac{1}{t(1+t^2)} = \frac{A}{t} + \frac{Bt+C}{1+t^2}.$$ [5]

(iii) Use integration, together with your results in parts **(i)** and **(ii)**, to show that

$$M = \frac{Kt}{\sqrt{1+t^2}},$$

where K is a constant. [6]

(iv) When $t = 1, M = 25$. Calculate K.

What is the mass of the chemical in the long term? [4]

8 Some years ago an island was populated by red squirrels and there were no grey squirrels. Then grey squirrels were introduced.

The population x, in thousands, of red squirrels is modelled by the equation

$$x = \frac{a}{1+kt},$$

where t is the time in years, and a and k are constants. When $t = 0$, $x = 2.5$.

(i) Show that $\dfrac{dx}{dt} = -\dfrac{kx^2}{a}$. [3]

(ii) Given that the initial population of 2.5 thousand red squirrels reduces to 1.6 thousand after one year, calculate a and k. [3]

(iii) What is the long-term population of red squirrels predicted by this model? [1]

The population y, in thousands, of grey squirrels is modelled by the differential equation

$$\frac{dy}{dt} = 2y - y^2.$$

When $t = 0$, $y = 1$.

(iv) Express $\dfrac{1}{2y - y^2}$ in partial fractions. [4]

(v) Hence show by integration that $\ln\left(\dfrac{y}{2-y}\right) = 2t$.

Show that $y = \dfrac{2}{1+e^{-2t}}$. [7]

(vi) What is the long-term population of grey squirrels predicted by this model? [1]

4 **(a)** The number of bacteria in a colony is increasing at a rate that is proportional to the square root of the number of bacteria present. Form a differential equation relating x, the number of bacteria, to the time t. [2]

(b) In another colony, the number of bacteria, y, after time t minutes is modelled by the differential equation

$$\frac{dy}{dt} = \frac{10000}{\sqrt{y}}.$$

Find y in terms of t, given that $y = 900$ when $t = 0$. Hence find the number of bacteria after 10 minutes. [6]

6 **(i)** Express $\dfrac{1}{(2x+1)(x+1)}$ in partial fractions. [3]

(ii) A curve passes through the point $(0, 2)$ and satisfies the differential equation

$$\frac{dy}{dx} = \frac{y}{(2x+1)(x+1)}.$$

Show by integration that $y = \dfrac{4x+2}{x+1}$. [5]

7 Data suggest that the number of cases of infection from a particular disease tends to oscillate between two values over a period of approximately 6 months.

(a) Suppose that the number of cases, P thousand, after time t months is modelled by the equation $P = \dfrac{2}{2 - \sin t}$. Thus, when $t = 0, P = 1$.

(i) By considering the greatest and least values of $\sin t$, write down the greatest and least values of P predicted by this model. [2]

(ii) Verify that P satisfies the differential equation $\dfrac{dP}{dt} = \frac{1}{2} P^2 \cos t$. [5]

(b) An alternative model is proposed, with differential equation

$$\frac{dP}{dt} = \tfrac{1}{2}(2P^2 - P)\cos t. \qquad (*)$$

As before, $P = 1$ when $t = 0$.

(i) Express $\dfrac{1}{P(2P-1)}$ in partial fractions. [4]

(ii) Solve the differential equation (*) to show that

$$\ln\left(\frac{2P-1}{P}\right) = \tfrac{1}{2}\sin t. \qquad [5]$$

This equation can be rearranged to give $P = \dfrac{1}{2 - e^{\frac{1}{2}\sin t}}$.

(iii) Find the greatest and least values of P predicted by this model. [4]

9 A skydiver drops from a helicopter. Before she opens her parachute, her speed $v\,\mathrm{m\,s^{-1}}$ after time t seconds is modelled by the differential equation

$$\frac{\mathrm{d}v}{\mathrm{d}t} = 10\mathrm{e}^{-\frac{1}{2}t}.$$

When $t = 0$, $v = 0$.

(i) Find v in terms of t. [4]

(ii) According to this model, what is the speed of the skydiver in the long term? [2]

She opens her parachute when her speed is $10\,\mathrm{m\,s^{-1}}$. Her speed t seconds after this is $w\,\mathrm{m\,s^{-1}}$, and is modelled by the differential equation

$$\frac{\mathrm{d}w}{\mathrm{d}t} = -\tfrac{1}{2}(w-4)(w+5).$$

(iii) Express $\dfrac{1}{(w-4)(w+5)}$ in partial fractions. [4]

(iv) Using this result, show that $\dfrac{w-4}{w+5} = 0.4\mathrm{e}^{-4.5t}$. [6]

(v) According to this model, what is the speed of the skydiver in the long term? [2]

7 Scientists can estimate the time elapsed since an animal died by measuring its body temperature.

(i) Assuming the temperature goes down at a constant rate of 1.5 degrees Fahrenheit per hour, estimate how long it will take for the temperature to drop

(*A*) from 98 °F to 89 °F,

(*B*) from 98 °F to 80 °F. **[2]**

In practice, rate of temperature loss is not likely to be constant. A better model is provided by Newton's law of cooling, which states that the temperature θ in degrees Fahrenheit t hours after death is given by the differential equation

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = -k(\theta - \theta_0),$$

where θ_0 °F is the air temperature and k is a constant.

(ii) Show by integration that the solution of this equation is $\theta = \theta_0 + A\mathrm{e}^{-kt}$, where A is a constant. **[5]**

The value of θ_0 is 50, and the initial value of θ is 98. The initial rate of temperature loss is 1.5 °F per hour.

(iii) Find A, and show that $k = 0.031\,25$. **[4]**

(iv) Use this model to calculate how long it will take for the temperature to drop

(*A*) from 98 °F to 89 °F,

(*B*) from 98 °F to 80 °F. **[5]**

(v) Comment on the results obtained in parts **(i)** and **(iv)**. **[1]**

3 A curve satisfies the differential equation $\frac{dy}{dx} = 3x^2y$, and passes through the point (1, 1). Find y in terms of x. **[4]**