6.3 METHODS FOR ADVANCED MATHEMATICS, C3 (4753) A2

Objectives

To build on and develop the techniques students have learnt at AS Level, with particular emphasis on the calculus.

Assessment

Examination	(72 marks) 1 hour 30 minutes The examination paper has two sections.		
	Section A:	5-7 questions, each worth at most 8 marks. Section Total: 36 marks	
	Section B:	two questions, each worth about 18 marks. Section Total: 36 marks	
Coursework	(18 marks)		

Candidates are required to undertake a piece of coursework on the numerical solution of equations (see pages 62 to 65).

Assumed Knowledge

Candidates are expected to know the content for Units *C1* and *C2*.

Subject Criteria

The Units *C1* and *C2* are required for Advanced Subsidiary GCE Mathematics in order to ensure coverage of the subject criteria.

The Units *C1, C2, C3* and *C4* are required for Advanced GCE Mathematics in order to ensure coverage of the subject criteria.

Calculators

In the MEI Structured Mathematics specification, no calculator is allowed in the examination for *C1*. For all other units, including this one, a graphical calculator is allowed.

METHODS FOR ADVANCED MATHEMATICS, C3		
Specification	**Ref.**	**Competence Statements**

PROOF		
Methods of proof.	C3p1	Understand, and be able to use, proof by direct argument, exhaustion and contradiction.
	2	Be able to disprove a conjecture by the use of a counter example.

EXPONENTIALS AND NATURAL LOGARITHMS		
The exponential and natural logarithm.	C3a1	Understand and be able to use the simple properties of exponential and logarithmic functions including the functions e^x and $\ln x$.
Functions.	2	Know the relationship between $\ln x$ and e^x.
	3	Know the graphs of $y = \ln x$ and $y = \mathrm{e}^x$.
	4	Be able to solve problems involving exponential growth and decay.

FUNCTIONS		
The language of functions.	C3f1	Understand the definition of a function, and the associated language.
	2	Know the effect of combined transformations on a graph and be able to form the equation of the new graph.
	3	Be able, given the graph of $y = \mathrm{f}(x)$, to sketch related graphs.
	4	Be able to apply transformations to the basic trigonometrical functions.
	5	Know how to find a composite function, $\mathrm{gf}(x)$.
	6	Know the conditions necessary for the inverse of a function to exist and how to find it (algebraically and graphically).
	7	Understand the functions arcsin, arccos and arctan, their graphs and appropriate restricted domains.
	8	Understand what is meant by the terms odd, even and periodic functions and the symmetries associated with them.
The modulus function.	9	Understand the modulus function.
	10	Be able to solve simple inequalities containing a modulus sign.

METHODS FOR ADVANCED MATHEMATICS, C3			
Ref.	**Notes**	**Notation**	**Exclusions**

PROOF

C3p1			
2			

EXPONENTIALS AND NATURAL LOGARITHMS

C3a1		$\log_e x = \ln x$	
2	Simplifying expressions involving exponentials and logarithms.		
3			
4			

FUNCTIONS

C3f1	Many-to-one, one-to-many, one-to-one, mapping, object, image, domain, codomain, range, odd, even, periodic.		
2	Translation parallel to the *x*-axis. Translation parallel to the *y*-axis. Stretch parallel to the *x*-axis. Stretch parallel to the *y*-axis. Reflection in the *x*-axis. Reflection in the *y*-axis. Combinations of these transformations.	Translation vector $\begin{pmatrix} a \\ b \end{pmatrix}$	
3	$y = \mathrm{f}(x \pm a)$ $\quad y = \mathrm{f}(x) \pm a$ $y = \mathrm{f}(ax)$ $\quad y = a\mathrm{f}(x)$ for $a > 0$ $y = \mathrm{f}(-x)$ $\quad y = -\mathrm{f}(x)$.		
4	Translations parallel to the *x*- and *y*-axes. Stretches parallel to the *x*- and *y*-axes. Reflections in the *x*- and *y*-axes.		
5			
6	The use of reflection in the line $y = x$. e.g. $\ln x$ ($x > 0$) is the inverse of e^x.		
7	Their graphs and periodicity.		
8	e.g. x^n for integer values of *n*.		
9	Graphs of linear functions involving a single modulus sign.		
10	Including the use of inequalities of the form $\lvert x - a \rvert \le b$ to express upper and lower bounds, $a \pm b$, for the value of *x*.		Inequalities involving more than one modulus sign.

METHODS FOR ADVANCED MATHEMATICS, C3		
Specification	**Ref.**	**Competence Statements**

CALCULUS

Specification	Ref.	Competence Statements
The product, quotient and chain rules.	C3c1	Be able to differentiate the product of two functions.
	2	Be able to differentiate the quotient of two functions.
	3	Be able to differentiate composite functions using the chain rule.
	4	Be able to find rates of change using the chain rule.
Inverse functions.	5	Be able to differentiate an inverse function.
Implicit differentiation.	6	Be able to differentiate a function defined implicitly.
Differentiation of further functions.	7	Be able to differentiate e^{ax} and $\ln x$.
	8	Be able to differentiate the trigonometrical functions: $\sin x$; $\cos x$; $\tan x$.
Integration by substitution.	9	Be able to use integration by substitution in cases where the process is the reverse of the chain rule.
	10	Be able to use integration by substitution in other cases.
Integration of further functions.	11	Be able to integrate $\frac{1}{x}$.
	12	Be able to integrate e^{ax}.
	13	Be able to integrate $\sin x$ and $\cos x$.
Integration by parts.	14	Be able to use the method of integration by parts in cases where the process is the reverse of the product rule.
	15	Be able to apply integration by parts to $\ln x$.

METHODS FOR ADVANCED MATHEMATICS, C3			
Ref.	**Notes**	**Notation**	**Exclusions**

CALCULUS

C3c1			
2			
3			
4			
5	$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)}$		
6	e.g. $\ln y = 1 - x^2$.		
7			
8	Including their sums and differences.		
9	e.g. $(1+2x)^8$, $x(1+x^2)^8$, xe^{x^2}, $\frac{1}{2x+3}$ Where appropriate, recognition may replace substitution.		
10	Simple cases only, e.g. $\frac{x}{2x+1}$.		
11			
12			
13			Integrals involving arcsin, arccos and arctan forms.
14	e.g. xe^x.		Integrals requiring more than one application of the method. Products of e^x and trigonometric functions.
15			

METHODS FOR ADVANCED MATHEMATICS, C3		
Specification	**Ref.**	**Competence Statements**

NUMERICAL METHODS

This topic will not be assessed in the examination for C3, since it is the subject of the coursework.

Change of sign.	C3e1	Be able to locate the roots of $\mathrm{f}(x)=0$ by considering changes of sign of $\mathrm{f}(x)$ in an interval of x in which $\mathrm{f}(x)$ is continuous.
	2	Be aware of circumstances under which change of sign methods may fail to give an expected root or may give a false root.
Fixed point iteration.	3	Be able to carry out a fixed point iteration after rearranging an equation into the form $x=\mathrm{g}(x)$.
	4	Understand that not all iterations converge to a particular root of an equation.
The Newton-Raphson method.	5	Be able to use the Newton-Raphson method to solve an equation.
Error Bounds.	6	Appreciate the need to establish error bounds when applying a numerical method.
Geometrical interpretation.	7	Be able to give a geometrical interpretation both of the processes involved and of their algebraic representation.

METHODS FOR ADVANCED MATHEMATICS, C3			
Ref.	**Notes**	**Notation**	**Exclusions**

NUMERICAL METHODS

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C3e1	e.g. decimal search.		
2	e.g. when the curve of $y = \mathrm{f}(x)$ touches the x-axis. e.g. when the curve of $y = \mathrm{f}(x)$ has a vertical asymptote.		
3	e.g. $x^3 - x - 4 = 0$ written as $x = \sqrt[3]{x+4}$ and so give rise to the iteration $x_{n+1} = \sqrt[3]{x_n + 4}$. Staircase and cobweb diagrams.		
4	The iteration $x_{n+1} = \mathrm{g}(x_n)$ converges to a root at $x = a$ if $\lvert \mathrm{g}'(a) \rvert < 1$ providing the iteration starts sufficiently close to a.		Proof.
5			
6	Error bounds should be established within the numerical method and not by reference to an already known solution.		
7			

Methods for Advanced Mathematics (C3) Coursework: Solution of Equations by Numerical Methods

Rationale

The assessment of this unit includes a coursework task (Component 2) involving the solution of equations by three different numerical methods.

The aims of this coursework are that students should appreciate the principles of numerical methods and at the same time be provided with useful equation solving techniques.

The objectives are:

- that students should be able to solve equations efficiently, to any required level of accuracy, using numerical methods;
- that in doing so they will appreciate how to use appropriate technology, such as calculators and computers, as a mathematical tool and have an awareness of its limitations;
- that they show geometrical awareness of the processes involved.

This task represents 20% of the assessment and the work involved should be consistent with that figure, both in quality and level of sophistication.

Numerical methods should be seen as complementing analytical ones and not as providing alternative (and less accurate) ways of doing the same job. Thus, equations which have simple analytical solutions should not be selected. Accuracy should be established from within the numerical working and not by reference to an exact solution obtained analytically.

The intention of this piece of coursework is not merely to solve equations; students should be encouraged to treat it as an investigation and to choose their own equations.

Requirements

1 Students must solve equations by the following three methods:

- Systematic search for a change of sign using one of the methods: bisection; decimal search; linear interpolation. One root is to be found.
- Fixed point iteration using the Newton-Raphson method. The equation selected must have at least two roots and all roots are to be found.
- Fixed point iteration after rearranging the equation $\mathrm{f}(x) = 0$ into the form $x = \mathrm{g}(x)$. One root is to be found.

A different equation must be used for each method.

In addition, a student's write-up must meet the following requirements.

2 One root of one of the equations must be found by all three methods. The methods used should then be compared in terms of their efficiency and ease of use.

3 The write-up must include graphical illustrations of how the methods work on the student's equations.

4 A student is expected to be able to give error bounds for the value of any root. This must be demonstrated in the case of the change of sign method (maximum possible error 0.5×10^{-3}), and for one of the roots found by the Newton-Raphson method (required accuracy five significant figures).

5 For each method an example should be given of an equation where the method fails: that is, an expected root is not obtained, a root is not found or a false root is obtained. There must be an explanation, illustrated graphically, of why this happens. In this situation it is acceptable to use equations with known analytical solutions provided they are not trivial.

Notation and Language

Students are expected to use correct notation and terminology. This includes distinguishing between the words function and equation, and between root and solution.

- For a *function* denoted by f(x), the corresponding *equation* is f(x) = 0. Thus the expression $x^3 - 3x^2 - 4x + 11$ is a function, $x^3 - 3x^2 - 4x + 11 = 0$ is an equation.
- The equation $x^3 - x = 0$ has three *roots*, namely x = -1, x = 0 and x = +1. The *solution* of the equation is x = -1, 0 or +1. Solving an equation involves finding all its roots.

Trivial Equations

Students should avoid trivial equations both when solving them, and where demonstrating failure. For an equation to be non-trivial it must pass two tests.

(i) It should be an equation they would expect to work on rather than just write down the solution (if it exists); for instance $\frac{1}{(x-a)} = 0$ is definitely not acceptable; nor is any polynomial expressed as a product of linear factors.

(ii) Constructing a table of values for integer values of x should not, in effect, solve the equation. Thus $x^3 - 6x^2 + 11x - 6 = 0$ (roots at x = 1, 2 and 3) is not acceptable.

Oral Communication

Each student must talk about the task; this may take the form of a class presentation, an interview with the assessor or ongoing discussion with the assessor while the work is in progress. Topics for discussion may include strategies used to find suitable equations and explanations, with reference to graphical illustrations, of how the numerical methods work.

Use of Software

The use of existing computer or calculator software is encouraged, but students must:

- edit any print-outs and displays to include only what is relevant to the task in hand;
- demonstrate understanding of what the software has done, and how they could have performed the calculations themselves;
- appreciate that the use of such software allows them more time to spend on investigational work.

Selection of Equations

Centres may provide students with a list of at least ten equations from which they can, if they wish, select those they are going to solve or use to demonstrate failure of a method. Such a list of equations should be forwarded to the Moderator with the sample of coursework requested. A new set of equations must be supplied with each examination series. Centres may, however, exercise the right not to issue a list, on the grounds that candidates stand to benefit from the mathematics they learn while finding their own equations.

Methods for Advanced Mathematics (C3) Coursework: Assessment Sheet

Task: Candidates will investigate the solution of equations using the following three methods:

- Systematic search for change of sign using one of the three methods: decimal search, bisection or linear interpolation.
- Fixed point iteration using the Newton-Raphson method.
- Fixed point iteration after rearranging the equation $\mathrm{f}(x)=0$ into the form $x=\mathrm{g}(x)$.

Coursework Title			
Candidate Name		**Candidate Number**	
Centre Number		**Date**	

Domain	Mark	Description	Comment	Mark
Change of sign method (3)	1 1 1	The method is applied successfully to find one root of an equation. Error bounds are stated and the method is illustrated graphically. An example is given of an equation where one of the roots cannot be found by the chosen method. There is an illustrated explanation of why this is the case.		
Newton-Raphson method (5)	1 1 1 1 1	The method is applied successfully to find one root of a second equation. All the roots of the equation are found. The method is illustrated graphically for one root. Error bounds are established for one of the roots. An example is given of an equation where this method fails to find a particular root despite a starting value close to it. There is an illustrated explanation of why this has happened.		
Rearranging f(x)=0 in the form x=g(x) (4)	1 1 1 1	A rearrangement is applied successfully to find a root of a third equation. Convergence of this rearrangement to a root is demonstrated graphically and the magnitude of $\mathrm{g}'(x)$ is discussed. A rearrangement of the same equation is applied in a situation where the iteration fails to converge to the required root. This failure is demonstrated graphically and the magnitude of $\mathrm{g}'(x)$ is discussed.		
Comparison of methods (3)	1 1 1	One of the equations used above is selected and the other two methods are applied successfully to find the same root. There is a sensible comparison of the relative merits of the three methods in terms of speed of convergence. There is a sensible comparison of the relative merits of the three methods in terms of ease of use with available hardware and software.		
Written communication (1)	1	Correct notation and terminology are used.		
Oral communication (2)	2	Presentation ☐ Interview ☐ Discussion ☐	Please tick at least one box and give a brief report.	
Half marks may be awarded but the overall total must be an integer. Please report overleaf on any help that the candidate has received beyond the guidelines			**TOTAL**	**18**

Coursework must be available for moderation by OCR