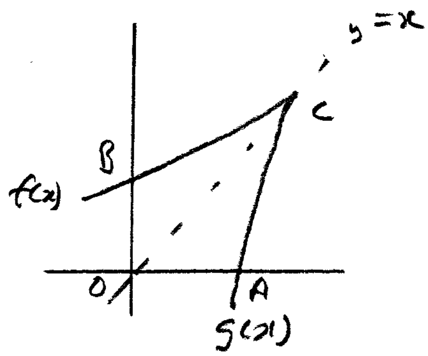
MEI CORE 3 JUNE 2009 SECTION B

8) i)

$$f(x) = e^{x-1}$$
$$g(x) = 1 + \ln x$$



At A

$$g(x) = 0 \Rightarrow 1 + \ln x = 0$$
$$\ln x = -1$$
$$x = e^{-1}$$

$A\left(\frac{1}{e}, 0\right)$

At B $\quad f(0) = e^{0-1} = e^{-1}$

$B\left(0, \frac{1}{e}\right)$

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ii) Let $y = f(x) = e^{x-1}$

swap variables $\quad x = e^{y-1}$

$$\ln x = y - 1$$
$$1 + \ln x = y \qquad y = g(x) \qquad \therefore g(x) \text{ is inverse of } f(x)$$

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iii)
$$\int_0^1 f(x)\,dx = \int_0^1 e^{x-1}\,dx = \left[e^{x-1}\right]_0^1$$
$$= e^{1-1} - e^{0-1}$$
$$= 1 - \frac{1}{e}$$

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i) Omitted verifying $C(1,1)$

When $x = 1$, $f(1) = e^{1-1} = e^0 = 1$

When $x = 1$ $\quad g(1) = 1 + \ln 1 = 1 + 0 = 1$

$\therefore$ curves intersect at $C(1,1)$

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9 iv) $\int \ln x \, dx = \int 1 \ln x \, dx$

Let $u = \ln x \Rightarrow \frac{du}{dx} = \frac{1}{x}$ Let $\frac{dv}{dx} = 1 \Rightarrow v = x$

$$\int u \frac{dv}{dx} = uv - \int v \frac{du}{dx}$$

$$\int \ln x \, dx = x \ln x - \int x \frac{1}{x} \, dx$$

$$= x \ln x - \int 1 \, dx$$

$$= x \ln x - x + c$$

$$\int_{e^{-1}}^{1} g(x) \, dx = \int_{e^{-1}}^{1} (1 + \ln x) \, dx = \left[x + x \ln x - x \right]_{e^{-1}}^{1}$$

$$= \left[x \ln x \right]_{e^{-1}}^{1}$$

$$= 1 \ln 1 - \frac{1}{e} \ln e^{-1}$$

$$= 0 + \frac{1}{e} \ln e = \frac{1}{e}$$

v) Area in shape OAC = Area in Δ under $y = x$ − answer to iii

$$= \frac{1}{2} \times 1 \times 1 - \frac{1}{e} = \frac{1}{2} - \frac{1}{e}$$

Symmetrical about $y = x$ so area of shape required

$$= 2\left(\frac{1}{2} - \frac{1}{e}\right)$$

$$= 1 - \frac{2}{e}$$

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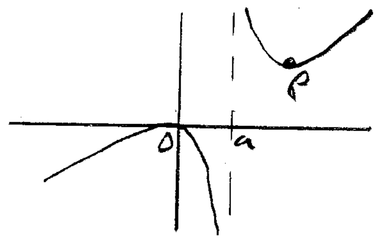
9) i) $y = \dfrac{x^2}{3x-1}$



$a = \dfrac{1}{3}$

ii) $\dfrac{dy}{dx} = \dfrac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} = \dfrac{(3x-1)(2x) - x^2(3)}{(3x-1)^2}$

$= \dfrac{6x^2 - 2x - 3x^2}{(3x-1)^2}$

$= \dfrac{3x^2 - 2x}{(3x-1)^2} = \dfrac{x(3x-2)}{(3x-1)^2}$

iii) At t.p $\dfrac{dy}{dx} = 0$ $\Rightarrow x(3x-2) = 0$

$\Rightarrow x = 0$ or $x = \frac{2}{3}$

At P $x = \frac{2}{3}$, $y = \dfrac{\left(\frac{2}{3}\right)^2}{3\left(\frac{2}{3}\right)-1} = \dfrac{\frac{4}{9}}{1} = \dfrac{4}{9}$

so $P\left(\frac{2}{3}, \frac{4}{9}\right)$

Gradient when $x = 0.6$ is $\dfrac{0.6(3(0.6)-2)}{(3(0.6)-1)^2} = -\dfrac{3}{16} < 0$

Gradient when $x = \frac{2}{3} = 0.\dot{6}$ is 0

Gradient when $x = 0.8$ is $\dfrac{0.8(3(0.8)-2)}{(3(0.8)-1)^2} = \dfrac{8}{49} > 0$

Gradient $- \backslash \underset{0}{\smile} / +$ passing through $x = \frac{2}{3}$

$\therefore$ a minimum

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9) iv) $\int \frac{x^2}{3x-1}\,dx$

Let $u = 3x-1$

$\frac{du}{dx} = 3$

$du = 3dx$

$\frac{1}{3}du = dx$

Also $u+1 = 3x$

$\frac{u+1}{3} = x$

$= \frac{1}{3}\int \frac{\left(\frac{u+1}{3}\right)^2}{u}\,du$

$= \frac{1}{3}\int \frac{u^2+2u+1}{9u}\,du$

$= \frac{1}{27}\int\left(u+2+\frac{1}{u}\right)du$

$x = \frac{2}{3} \quad u = 1$

$x = 1 \quad u = 2$

$\int_{2/3}^{1} \frac{x^2}{3x-1}\,dx = \frac{1}{27}\int_1^2\left(u+2+\frac{1}{u}\right)du$

$= \frac{1}{27}\left[\frac{u^2}{2}+2u+\ln u\right]_1^2$

$= \frac{1}{27}\left[(2+4+\ln 2)-\left(\frac{1}{2}+2+\ln 1\right)\right]$

$= \frac{1}{27}\left[6+\ln 2 - 2\tfrac{1}{2}\right]$

$= \frac{1}{27}\left[\frac{7}{2}+\ln 2\right] = \frac{7+2\ln 2}{54}$