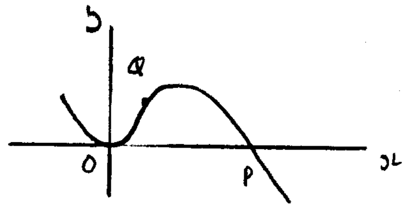
MEI CORE 3 JUNE 2005 SECTION B

8) 

$y = x \sin 3x$

i) At P, $y = 0 \Rightarrow x \sin 3x = 0$

$\Rightarrow x = 0$ or $\sin 3x = 0$

$\Rightarrow 3x = 0$ or $3x = \pi$

$x = \frac{\pi}{3}$

$P\left(\frac{\pi}{3}, 0\right)$

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ii) At Q we are given $x = \frac{\pi}{6}$

$\Rightarrow y = \frac{\pi}{6} \sin \frac{3\pi}{6} = \frac{\pi}{6} \times 1 = \frac{\pi}{6}$

$Q\left(\frac{\pi}{6}, \frac{\pi}{6}\right)$ is therefore on line $y = x$

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iii) $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} = x \times 3\cos 3x + \sin 3x \times 1$

$= 3x\cos 3x + \sin 3x$

When $x = \frac{\pi}{6}$, $\frac{dy}{dx} = 3 \times \frac{\pi}{6} \times \cos\frac{3\pi}{6} + \sin\frac{3\pi}{6}$

$= 0 + 1$

$= 1$

This is same gradient as $y = x$

$\therefore y = x$ is a tangent to curve at $Q\left(\frac{\pi}{6}, \frac{\pi}{6}\right)$

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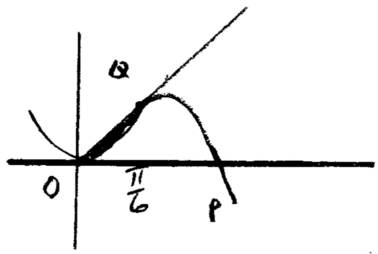
iv) 

Area between line and curve

= Area under line $y = x$

− Area under curve

between $x = 0$ and $x = \frac{\pi}{6}$

8iv) cont. Area under $y=x$ is a triangle

$$\text{Area} = \tfrac{1}{2}\,\text{base} \times \text{height} = \frac{1}{2} \times \frac{\pi}{6} \times \frac{\pi}{6} = \frac{\pi^2}{72}$$

$$\text{Area under curve} = \int_0^{\frac{\pi}{6}} x \sin 3x \, dx$$

Integration by Parts

Let $u = x$ $\Rightarrow \frac{du}{dx} = 1$

Let $\frac{dv}{dx} = \sin 3x$ $\Rightarrow v = -\frac{1}{3}\cos 3x$

$$\int u\frac{dv}{dx} = uv - \int v\frac{du}{dx}$$

$$\int_0^{\frac{\pi}{6}} x \sin 3x \, dx = \left[-\frac{1}{3}x\cos 3x\right]_0^{\frac{\pi}{6}} + \int_0^{\frac{\pi}{6}} \frac{1}{3}\cos 3x \, dx$$

$$= \left[-\frac{1}{3}x\cos 3x\right]_0^{\frac{\pi}{6}} + \left[\frac{1}{9}\sin 3x\right]_0^{\frac{\pi}{6}}$$

$$= \left(-\frac{1}{3} \times \frac{\pi}{6}\cos\frac{\pi}{2}\right) - \left(\frac{1}{3} \times 0\cos 0\right) + \left(\frac{1}{9}\sin\frac{\pi}{2}\right) - \left(\frac{1}{9}\sin 0\right)$$

$$= 0 - 0 + \frac{1}{9} - 0$$

$$= \frac{1}{9}$$

$\therefore$ area between line $y=x$ and curve

$$= \frac{\pi^2}{72} - \frac{1}{9}$$

$$= \frac{1}{72}\left(\pi^2 - 8\right)$$

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9) $f(x) = \ln(1+x^2)$ for $-3 \le x \le 3$

i) $f(-x) = \ln(1+(-x)^2) = \ln(1+x^2) = f(x)$

$\therefore$ f is an even function

As an even function it is symmetrical about the y-axis

ii) $\frac{d}{dx}\ln(1+x^2) = \frac{2x}{1+x^2}$

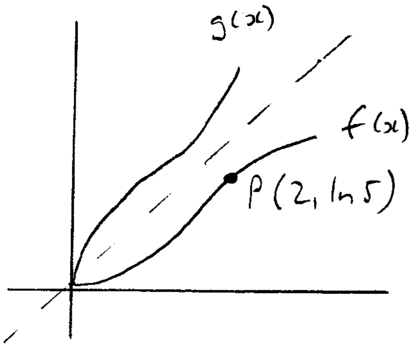
When $x = 2$, $f'(2) = \frac{2 \times 2}{1+2^2} = \frac{4}{5}$

At $P(2, \ln 5)$ gradient $= \frac{4}{5}$

iii) f does not have an inverse for domain $-3 \le x \le 3$

because it is not 1 to 1

iv)



Domain of $g(x)$ is $x \geqslant 0$

Let $y = \ln(1+x^2)$

Swap variables $x = \ln(1+y^2)$

$e^x = 1+y^2$

$e^x - 1 = y^2$

$\sqrt{e^x - 1} = y$

$\therefore g(x) = \sqrt{e^x - 1}$

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9 v)

$$g(x) = \sqrt{e^x - 1} = (e^x - 1)^{\frac{1}{2}}$$

$$g'(x) = \tfrac{1}{2}(e^x - 1)^{-\frac{1}{2}} \times e^x$$

$$= \frac{e^x}{2\sqrt{e^x - 1}}$$

$$g'(\ln 5) = \frac{e^{\ln 5}}{2\sqrt{e^{\ln 5} - 1}}$$

$$= \frac{5}{2\sqrt{5-1}}$$

$$= \frac{5}{4} \qquad = 1\tfrac{1}{4}$$

This is the positive reciprocal of the result found in part ii

The gradient of $g(x)$ at $(\ln 5, 2)$
is the positive reciprocal of the gradient of $f(x)$ at $(2, \ln 5)$