MEI CORE 3 JANUARY 2008 SECTION B

7) i) $y = 2x\ln(1+x)$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} = 2x.\frac{1}{1+x} + \ln(1+x) \times 2$$

$$\frac{dy}{dx} = \frac{2x}{1+x} + 2\ln(1+x)$$

At st.pt. $\frac{dy}{dx} = 0$ When $x = 0$, $y = 2 \times 0 \times \ln 1 = 0$

$\therefore$ $(0,0)$ on curve

When $x = 0$, $\frac{dy}{dx} = \frac{0}{1} + 2\ln 1 = 0$

$\therefore$ st. pt at origin

ii) $$\frac{d^2y}{dx^2} = \frac{(1+x)2 - 2x(1)}{(1+x)^2} + \frac{2}{1+x}$$

$$= \frac{2 + 2x - 2x}{(1+x)^2} + \frac{2}{1+x}$$

$$= \frac{2}{(1+x)^2} + \frac{2}{1+x}$$

When $x = 0$, $\frac{d^2y}{dx^2} = \frac{2}{1^2} + \frac{2}{1} = 4 > 0$

$\therefore$ minimum at $(0,0)$

iii) $$\int \frac{x^2}{1+x}\,dx$$

Let $u = 1+x$

$\frac{du}{dx} = 1$

$du = dx$

Also $x = u - 1$

MEI CORE 3 — JANUARY 2008 — SECTION B

7iii) cont)

$$= \int \frac{(u-1)^2}{u}\,du = \int \frac{u^2-2u+1}{u}\,du$$

$$= \int \left(u - 2 + \frac{1}{u}\right) du$$

$$\int_0^1 \frac{x^2}{1+x}\,dx$$

when $x = 1$ $u = 2$
$x = 0$ $u = 1$

$$= \int_1^2 \left(u - 2 + \frac{1}{u}\right) du = \left[\frac{u^2}{2} - 2u + \ln u\right]_1^2$$

$$= \left(\frac{2^2}{2} - 2(2) + \ln 2\right) - \left(\frac{1}{2} - 2 + \ln 1\right)$$

$$= -2 + \ln 2 - \left(-\frac{3}{2}\right)$$

$$= -\frac{1}{2} + \ln 2$$

7iv) $\int_0^1 2x\ln(1+x)$

Let $u = \ln(1+x)$ $\quad \frac{du}{dx} = \frac{1}{1+x}$

Let $\frac{dv}{dx} = 2x$ $\quad v = x^2$

$$\int_0^1 2x\ln(1+x) = \left[x^2\ln(1+x)\right]_0^1 - \int_0^1 \frac{x^2}{1+x}\,dx$$

$$= \left[(1\ln 2 - 0)\right] - \left(-\frac{1}{2} + \ln 2\right)$$

$$= \ln 2 + \frac{1}{2} - \ln 2$$

$$= \frac{1}{2}$$

8) i) $y = f(x) = 1 + \sin 2x \qquad -\frac{\pi}{4} \le x \le \frac{\pi}{4}$

One way stretch by scale factor $\frac{1}{2}$ parallel to x-axis
Translation by $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Transformations could be carried out in either order

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ii) $$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (1 + \sin 2x)\,dx = \left[x - \tfrac{1}{2}\cos 2x \right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$$

$$= \left(\frac{\pi}{4} - \frac{1}{2}\cos\frac{\pi}{2} \right) - \left(-\frac{\pi}{4} - \frac{1}{2}\cos\left(-\frac{\pi}{2}\right) \right)$$

$$= \frac{\pi}{4} - 0 + \frac{\pi}{4} + 0 = \frac{\pi}{2}$$

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iii) $f'(x) = 2\cos 2x$

when $x = 0$, $f'(0) = 2\cos 0 = 2$

Gradient of $f(x)$ at $(0,1)$ is 2

Gradient of $f^{-1}(x)$ at $(1,0)$ is $\frac{1}{2}$

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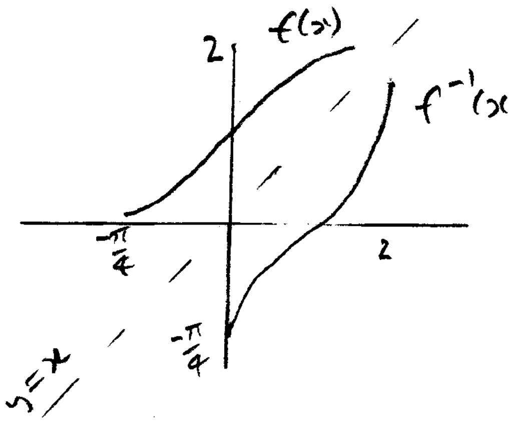
iv) Domain of $f^{-1}(x)$ is same as range of $f(x)$

Domain of $f^{-1}(x)$ is $0 \le x \le 2$



8 v)

Let $y = 1 + \sin 2x$

Swap variables

$$x = 1 + \sin 2y$$

$$x - 1 = \sin 2y$$

$$\sin^{-1}(x-1) = 2y$$

$$\frac{\sin^{-1}(x-1)}{2} = y$$

$$f^{-1}(x) = \frac{\sin^{-1}(x-1)}{2}$$