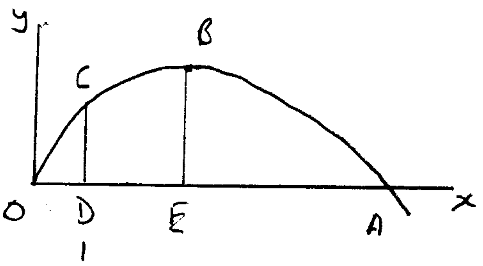
MEI CORE 3 JAN 2006 SECTION B

7) i) $y = 2x - x\ln x$



At A, $y = 0$

$2x - x\ln x = 0$

$x(2 - \ln x) = 0$ At A $2 - \ln x = 0$

$2 = \ln x$

$e^2 = x$

At A, x coord $= e^2$

ii) $\frac{dy}{dx} = 2 - \left(x\frac{1}{x} + \ln x \times 1\right) = 2 - 1 - \ln x$

$= 1 - \ln x$

At t.p B $\frac{dy}{dx} = 0 \Rightarrow 1 - \ln x = 0$

$1 = \ln x$

$e = x$

When $x = e$, $y = 2e - e\ln e$

$= 2e - e$

$= e$

$\therefore B(e, e)$

iii) At $A(e^2, 0)$ $\frac{dy}{dx} = 1 - \ln(e^2) = 1 - 2\ln e = 1 - 2 = -1$

At C, $x = 1 \Rightarrow \frac{dy}{dx} = 1 - \ln 1 = 1 - 0 = 1$

Since $1 \times -1 = -1$ gradients of curve and $\therefore$ of tangents at A and C are perpendicular

MEI CORE 3 JAN 2006 SECTION B

7iv) $\int x \ln x \, dx$

Let $u = \ln x$ $\Rightarrow \frac{du}{dx} = \frac{1}{x}$

Let $\frac{dv}{dx} = x$ $\Rightarrow v = \frac{x^2}{2}$

$$\int u \frac{dv}{dx} = uv - \int v \frac{du}{dx}$$

$$\int x \ln x \, dx = \frac{x^2}{2} \ln x - \int \frac{x^2}{2} \cdot \frac{1}{x} \, dx$$

$$= \frac{x^2}{2} \ln x - \int \frac{x}{2} \, dx$$

$$= \frac{x^2}{2} \ln x - \frac{x^2}{4} + c$$

$$\text{Area} = \int_1^e (2x - x \ln x) \, dx$$

$$= \int_1^e 2x \, dx - \int_1^e x \ln x \, dx$$

$$= \left[x^2\right]_1^e - \left[\frac{x^2}{2} \ln x - \frac{x^2}{4}\right]_1^e$$

$$= e^2 - 1 - \left[\left(\frac{e^2}{2} \ln e - \frac{e^2}{4}\right) - \left(\frac{1^2}{2} \ln 1 - \frac{1}{4}\right)\right]$$

$$= e^2 - 1 - \left[\frac{e^2}{2} - \frac{e^2}{4} + 0 + \frac{1}{4}\right]$$

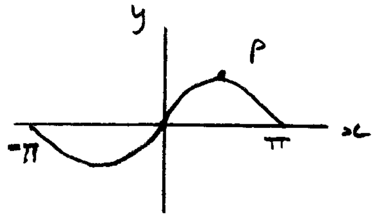
$$= \frac{3e^2}{4} - \frac{5}{4}$$

MEI CORE 3 JAN 2006 SECTION B

8) $$f(x) = \frac{\sin x}{2-\cos x} \quad \text{for} \quad -\pi \le x \le \pi$$

i) $$f(-x) = \frac{\sin(-x)}{2-\cos(-x)} = \frac{-\sin x}{2-\cos x} = -\frac{\sin x}{2-\cos x} = -f(x)$$

odd function which has rotational symmetry of order 2 about origin



ii) $$f'(x) = \frac{(2-\cos x)\cos x - \sin x(\sin x)}{(2-\cos x)^2}$$

$$f'(x) = \frac{2\cos x - \cos^2 x - \sin^2 x}{(2-\cos x)^2}$$

$$f'(x) = \frac{2\cos x - (\cos^2 x + \sin^2 x)}{(2-\cos x)^2} = \frac{2\cos x - 1}{(2-\cos x)^2}$$

At t.p. P, $f'(x) = 0$

$$\Rightarrow \quad 2\cos x - 1 = 0$$
$$2\cos x = 1$$
$$\cos x = \frac{1}{2}$$
$$x = \frac{\pi}{3}$$

When $x = \frac{\pi}{3}$, $$f\left(\frac{\pi}{3}\right) = \frac{\sin\frac{\pi}{3}}{2-\cos\frac{\pi}{3}} = \frac{\frac{\sqrt{3}}{2}}{2-\frac{1}{2}} = \frac{\frac{\sqrt{3}}{2}}{\frac{3}{2}} = \frac{1}{\sqrt{3}}$$

$\therefore \quad P\left(\frac{\pi}{3}, \frac{1}{\sqrt{3}}\right)$

Range of $f(x)$ $$-\frac{1}{\sqrt{3}} \le f(x) \le \frac{1}{\sqrt{3}}$$

MEI CORE 3 JAN 2006 SECTION B

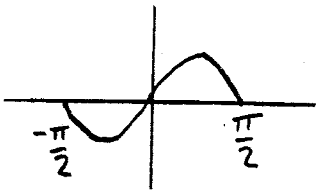
8 iii) $\displaystyle\int_0^{\pi} \frac{\sin x}{2-\cos x}\,dx = \Big[\ln(2-\cos x)\Big]_0^{\pi}$ (since numerator is differential of denominator)

$= \ln(2-\cos\pi) - \ln(2-\cos 0)$

$= \ln 3 - \ln 1$

$= \ln 3$

iv)



Stretch by scale factor $\frac{1}{2}$ parallel to x-axis

v) Area is half of previous area

so integral is half of previous integral

$$\int_0^{\frac{\pi}{2}} \frac{\sin 2x}{2-\cos 2x}\,dx = \frac{1}{2}\ln 3$$