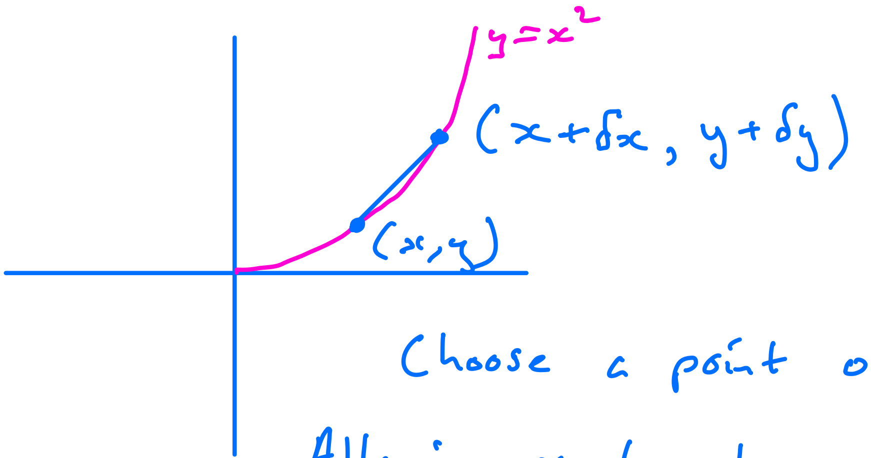
This is concerned with rates of change

Differentiate $y = x^2$ from first principles

Find the gradient at any point on the curve $y = x^2$



Choose a point on curve (x, y)

Allowing x to change by a small quantity δx causes a small change in y say δy

The chord from (x, y) to $(x + \delta x, y + \delta x)$ is an approximation to the tangent to the curve at (x, y)

Its gradient $= \dfrac{y + \delta y - y}{x + \delta x - x} = \dfrac{\delta y}{\delta x}$

As we let $\delta x \to 0$ then $\delta y \to 0$ and the chord tends to the tangent at (x, y)

Since both points on curve

$$y = x^2 \qquad (1)$$

$$y + \delta y = (x + \delta x)^2$$

$$y + \delta y = x^2 + 2x\,\delta x + (\delta x)^2 \qquad (2)$$

$(2) - (1)$

$$\delta y = 2x\,\delta x + (\delta x)^2$$

$\div\, \delta x$

$$\frac{\delta y}{\delta x} = \frac{2x\,\delta x}{\delta x} + \frac{(\delta x)^2}{\delta x}$$

$$\frac{\delta y}{\delta x} = 2x + \delta x$$

We call the limit as $\delta x \to 0$ of $\frac{\delta y}{\delta x}$ equal to $\frac{dy}{dx}$

$$\underset{\text{limit } \delta x \to 0}{\frac{\delta y}{\delta x}} = 2x + 0$$

$$\frac{dy}{dx} = 2x$$

$\therefore$ the gradient at any point (x, y) on the curve $y = x^2$ is given by $2x$

Alternative Notations

$y = x^2$

$\frac{dy}{dx} = 2x$

$\frac{d}{dx} x^2 = 2x$

$f(x) = x^2$

$f'(x) = 2x$

In general $\quad \frac{d}{dx} x^n = n x^{n-1}$

$$\frac{d}{dx} a x^n = n a x^{n-1}$$

for any constant a

for all rational powers n

Examples

$$\frac{d}{dx} x^4 = 4x^3$$

$$\frac{d}{dx} x^{10} = 10x^9$$

$$\frac{d}{dx} x = 1x^0 = 1$$

$$\frac{d}{dx} c = 0$$

for later

$$\frac{d}{dx} \frac{1}{x} = \frac{d}{dx} x^{-1} = -1x^{-2} = \frac{-1}{x^2}$$

$$\frac{d}{dx} \sqrt{x} = \frac{d}{dx} x^{\frac{1}{2}} = \frac{1}{2} x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$

Exercise Find $\frac{dy}{dx}$

1) $y = 3x^2 + 5x - 7$

$\frac{dy}{dx} = 6x + 5$

2) $y = 7x^7 - 6x^6 + 3x^3 + 1$

$\frac{dy}{dx} = 49x^6 - 36x^5 + 9x^2$

3) $y = x^4 + x^3 + x^2 + x + 1$

$\frac{dy}{dx} = 4x^3 + 3x^2 + 2x + 1$

4) $y = \frac{1}{2}x^4 - \frac{1}{3}x^3 - \frac{1}{4}x^2 + \frac{1}{5}x - \frac{1}{6}$

$\frac{dy}{dx} = \frac{4}{2}x^3 - \frac{3}{3}x^2 - \frac{2}{4}x + \frac{1}{5}$

$\frac{dy}{dx} = 2x^3 - x^2 - \frac{1}{2}x + \frac{1}{5}$

Typical Question

Find the equation of the tangent to the curve $y = x^3 - x^2 - 5x + 3$ at the point where $x = 2$

and find the equation of the normal to the curve when $x = 4$

Solution when $x = 2$

$$y = 2^3 - 2^2 - 5(2) + 3$$

$$y = 8 - 4 - 10 + 3 = -3$$

Point is $(2, -3)$

$$\frac{dy}{dx} = 3x^2 - 2x - 5$$

when $x = 2$ $\frac{dy}{dx} = 3(2)^2 - 2(2) - 5$

$$= 12 - 4 - 5$$

$$= 3$$

Gradient of tgt is therefore 3

$$y - y_1 = m(x - x_1)$$

$$y - -3 = 3(x - 2)$$

$$y + 3 = 3x - 6$$

$$y = 3x - 6 - 3$$

Tangent is $\underline{y = 3x - 9}$

Find eqn of normal when $x = 4$

when $x = 4$ $y = 4^3 - 4^2 - 5(4) + 3$

$$y = 64 - 16 - 20 + 3$$

$y = 31$

Point on curve $(4, 31)$

When $x = 4$ $\quad \frac{dy}{dx} = 3(4)^2 - 2(4) - 5$

$= 48 - 8 - 5$

$= 35$

gradient of tgt $= 35$

$\Rightarrow$ gradient of normal $= -\frac{1}{35}$

$y - y_1 = m(x - x_1)$

$y - 31 = -\frac{1}{35}(x - 4)$

$35y - 1085 = -x + 4$

Normal $\quad \underline{x + 35y - 1089 = 0}$