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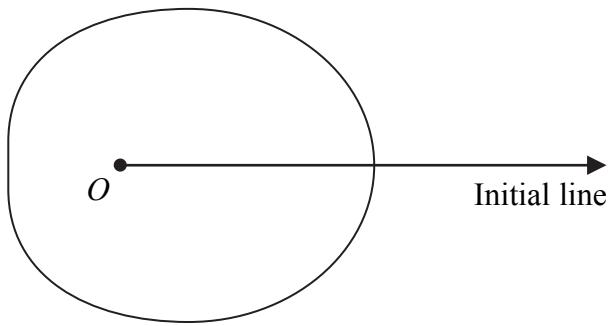


Figure 1

Figure 1 shows a sketch of the curve with polar equation

$$r = a + 3 \cos \theta, \quad a > 0, \quad 0 \leq \theta < 2\pi$$

The area enclosed by the curve is $\frac{107}{2} \pi$.

Find the value of a .

$$\begin{aligned}
 \text{Area} &= \int_0^{2\pi} \frac{1}{2} r^2 d\theta && (8) \\
 &= \frac{1}{2} \int_0^{2\pi} (a + 3\cos\theta)^2 d\theta \\
 &= \frac{1}{2} \int_0^{2\pi} (a^2 + 6a\cos\theta + 9\cos^2\theta) d\theta \\
 &= \frac{1}{2} \int_0^{2\pi} \left(a^2 + 6a\cos\theta + \frac{9}{2}(\cos 2\theta + 1) \right) d\theta \\
 &= \frac{1}{2} \int_0^{2\pi} \left(a^2 + \frac{9}{2} + 6a\cos\theta + \frac{9}{2}\cos 2\theta \right) d\theta \\
 &= \frac{1}{2} \left[a^2\theta + \frac{9\theta}{2} + 6a\sin\theta + \frac{9}{4}\sin 2\theta \right]_0^{2\pi}
 \end{aligned}$$

Question 4 continued

$$= \frac{1}{2} \left[(2\pi a^2 + 9\pi + 0 + 0) - (0) \right]$$

$$= \pi a^2 + \frac{9\pi}{2}$$

$$\text{Given area} = \frac{107\pi}{2}$$

$$\Rightarrow \pi a^2 + \frac{9}{2}\pi = \frac{107\pi}{2}$$

$$\pi a^2 = \frac{98\pi}{2}$$

$$a^2 = 49$$

$$a = 7 \quad (\text{since } a > 0)$$



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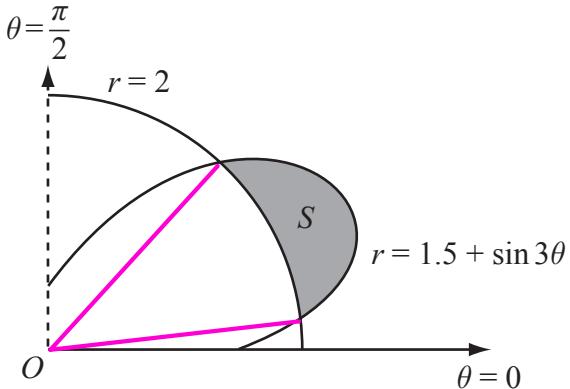
**Figure 1**

Figure 1 shows the curves given by the polar equations

$$r = 2, \quad 0 \leq \theta \leq \frac{\pi}{2},$$

$$\text{and} \quad r = 1.5 + \sin 3\theta, \quad 0 \leq \theta \leq \frac{\pi}{2}.$$

- (a) Find the coordinates of the points where the curves intersect.

(3)

The region S, between the curves, for which $r > 2$ and for which $r < (1.5 + \sin 3\theta)$, is shown shaded in Figure 1.

- (b) Find, by integration, the area of the shaded region S, giving your answer in the form $a\pi + b\sqrt{3}$, where a and b are simplified fractions.

(7)

a) $2 = 1.5 + \sin 3\theta$

$$0.5 = \sin 3\theta$$

$$3\theta = \sin^{-1} 0.5 = \frac{\pi}{6}, \frac{5\pi}{6}$$

$$\theta = \frac{\pi}{18}, \frac{5\pi}{18}$$

Intersect at $(2, \frac{\pi}{18})$ and $(2, \frac{5\pi}{18})$



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Question 5 continued

$$\text{Let } r_1 = 2, \quad r_2 = 1.5 + \sin 3\theta$$

$$\text{Shaded area } S = \int_{\frac{\pi}{18}}^{\frac{5\pi}{18}} \frac{1}{2} r_2^2 d\theta - \int_{\frac{\pi}{18}}^{\frac{5\pi}{18}} \frac{1}{2} r_1^2 d\theta$$

$$= \frac{1}{2} \int_{\frac{\pi}{18}}^{\frac{5\pi}{18}} (2.25 + 3 \sin 3\theta + \sin^2 3\theta) d\theta - \int_{\frac{\pi}{18}}^{\frac{5\pi}{18}} 2 d\theta$$

$$= \frac{1}{2} \int_{\frac{\pi}{18}}^{\frac{5\pi}{18}} \left(2.25 + 3 \sin 3\theta + \frac{1 - \cos 6\theta}{2} \right) d\theta - [2\theta]_{\frac{\pi}{18}}^{\frac{5\pi}{18}}$$

$$= \frac{1}{2} \left[2.25\theta - \cos 3\theta + \frac{1}{2}\theta - \frac{\sin 6\theta}{12} \right]_{\frac{\pi}{18}}^{\frac{5\pi}{18}} - \frac{8\pi}{18}$$

$$= \frac{1}{2} \left[\left(\frac{5\pi}{8} + \frac{\sqrt{3}}{2} + \frac{5\pi}{36} + \frac{\sqrt{3}}{24} \right) - \left(\frac{\pi}{8} - \frac{\sqrt{3}}{2} + \frac{\pi}{36} - \frac{\sqrt{3}}{24} \right) \right] - \frac{8\pi}{18}$$

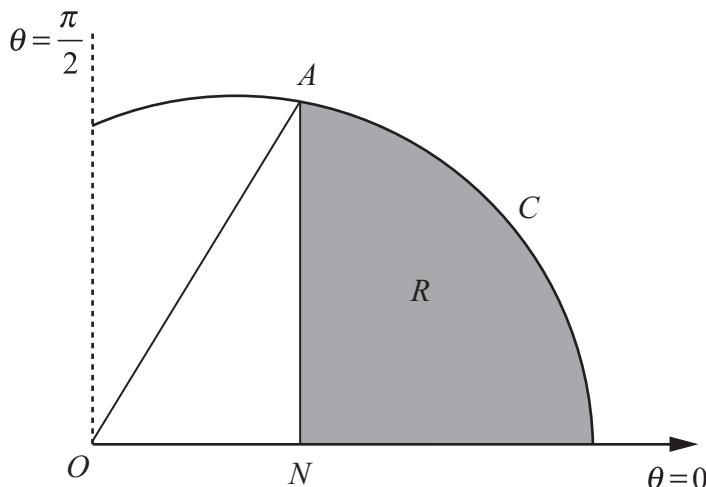
$$= \frac{1}{2} \left[\frac{11\pi}{18} + \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{12} \right] - \frac{8\pi}{18}$$

$$= \frac{11\pi}{36} + \frac{13\sqrt{3}}{24} - \frac{16\pi}{36}$$

$$= \frac{13\sqrt{3}}{24} - \frac{5\pi}{36}$$



6.

**Figure 1**

The curve C shown in Figure 1 has polar equation

$$r = 2 + \cos \theta, \quad 0 \leq \theta \leq \frac{\pi}{2}$$

At the point A on C , the value of r is $\frac{5}{2}$.

The point N lies on the initial line and AN is perpendicular to the initial line.

The finite region R , shown shaded in Figure 1, is bounded by the curve C , the initial line and the line AN .

Find the exact area of the shaded region R .

(9)

At A $\frac{5}{2} = 2 + \cos \theta$

$$\frac{1}{2} = \cos \theta \Rightarrow \theta = \frac{\pi}{3}$$

$$ON = r \cos \theta = \frac{5}{2} \cos \frac{\pi}{3} = \frac{5}{4}$$

$$\text{Area of } \triangle AON = \frac{1}{2} ab \sin C$$

$$= \frac{1}{2} \times \frac{5}{4} \times \frac{5}{2} \sin \frac{\pi}{3}$$

$$= \frac{25\sqrt{3}}{32}$$



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Question 6 continued

$$\text{Area traced out by curve} = \int_0^{\frac{\pi}{3}} \frac{1}{2} r^2 d\theta$$

$$= \frac{1}{2} \int_0^{\frac{\pi}{3}} (2 + \cos\theta)^2 d\theta$$

$$= \frac{1}{2} \int_0^{\frac{\pi}{3}} (4 + 4\cos\theta + \cos^2\theta) d\theta$$

$$= \frac{1}{2} \int_0^{\frac{\pi}{3}} \left(4 + 4\cos\theta + \frac{1 + \cos 2\theta}{2} \right) d\theta$$

$$= \frac{1}{2} \left[\frac{9}{2}\theta + 4\sin\theta + \frac{1}{4}\sin 2\theta \right]_0^{\frac{\pi}{3}}$$

$$= \frac{1}{2} \left[\left(\frac{9\pi}{6} + 4 \times \frac{\sqrt{3}}{2} + \frac{1}{4} \times \frac{\sqrt{3}}{2} \right) - 0 \right]$$

$$= \frac{3\pi}{4} + \sqrt{3} + \frac{\sqrt{3}}{16}$$

$$= \frac{3\pi}{4} + \frac{17\sqrt{3}}{16}$$

$$\text{Region R} = \frac{3\pi}{4} + \frac{17\sqrt{3}}{16} - \Delta A_{\text{on}}$$

$$= \frac{3\pi}{4} + \frac{17\sqrt{3}}{16} - \frac{25\sqrt{3}}{32} = \frac{3\pi}{4} + \frac{9\sqrt{3}}{32}$$



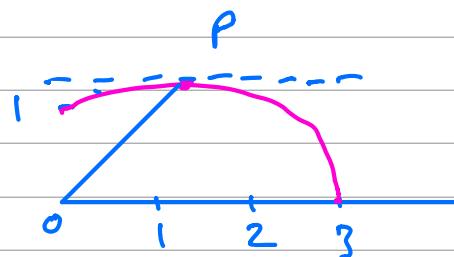
2. The curve C has polar equation

$$r = 1 + 2 \cos \theta, \quad 0 \leq \theta \leq \frac{\pi}{2}$$

At the point P on C , the tangent to C is parallel to the initial line.

Given that O is the pole, find the exact length of the line OP .

(7)



Let $P(x, y)$

$$x = r \cos \theta$$

$$y = r \sin \theta$$

$$y = (1 + 2 \cos \theta) \sin \theta$$

$$y = \sin \theta + 2 \sin \theta \cos \theta$$

$$y = \sin \theta + \sin 2\theta$$

$$\text{At } P, \frac{dy}{dx} = 0$$

$$\frac{dy}{d\theta} = \cos \theta + 2 \cos 2\theta$$

$$\Rightarrow \frac{dy}{d\theta} = 0$$

$$\text{At } P, \cos \theta + 2 \cos 2\theta = 0$$

$$\cos \theta + 4 \cos^2 \theta - 2 = 0$$

$$4 \cos^2 \theta + \cos \theta - 2 = 0$$

$$\cos \theta = \frac{-1 \pm \sqrt{33}}{8}$$

$$\cos \theta = \frac{-1 + \sqrt{33}}{8}$$



$$\begin{aligned}OP &= r = 1 + 2 \cos \theta \\&= 1 + \frac{-1 + \sqrt{33}}{4} \\&= \frac{3 + \sqrt{33}}{4}\end{aligned}$$
