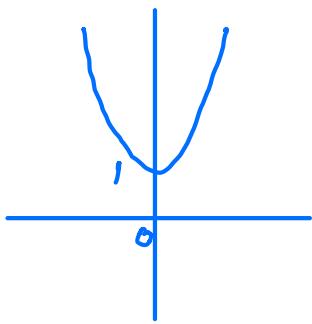
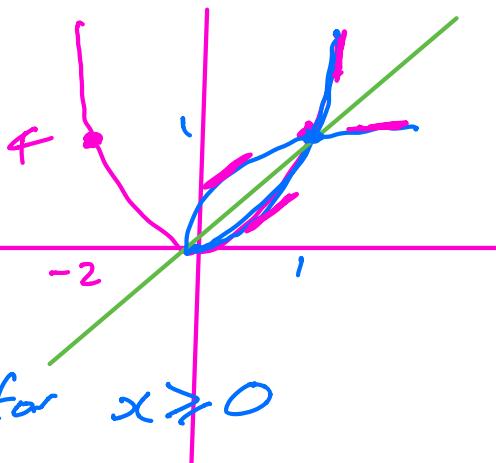


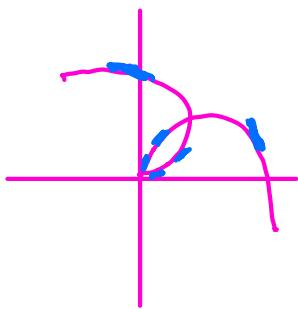
# Inverse Hyperbolic Functions



$$y = \cosh x$$



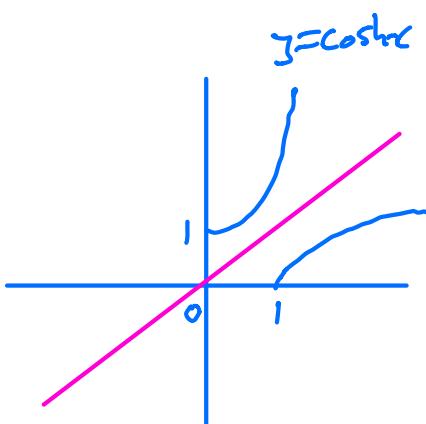
Inverse Functions in General



$$f(x) = x^2 \quad \text{for } x \geq 0$$

$$f^{-1}(x) = \sqrt{x} \quad \text{for } x \geq 0$$

# Inverse Hyperbolic Functions



$$y = \cosh x$$

$$y = \cosh^{-1} x$$

$$y = \cosh^{-1} x$$

$$\Rightarrow \cosh y = x$$

$$\frac{1}{2} (e^y + e^{-y}) = x$$

$$e^y + e^{-y} = 2x$$

$$xe^y$$

$$e^{2y} + 1 = 2xe^y$$

$$e^{2y} - 2xe^y + 1 = 0$$

$$e^y = \frac{2x \pm \sqrt{4x^2 - 4}}{2}$$

$$e^y = \frac{2x \pm 2\sqrt{x^2 - 1}}{2}$$

$$e^y = x \pm \sqrt{x^2 - 1}$$

$$y = \ln(x \pm \sqrt{x^2 - 1})$$

$$y = \underline{\ln(x + \sqrt{x^2 - 1})} \text{ or } \underline{\ln(x - \sqrt{x^2 - 1})}$$

Sum of roots  $\ln((x + \sqrt{x^2 - 1})(x - \sqrt{x^2 - 1}))$

$$= \ln(x^2 - (x^2 - 1))$$
$$= \ln 1 = 0$$

$\operatorname{arcosh}^{-1} x > 0 \quad \therefore y = \ln(x + \sqrt{x^2 - 1})$

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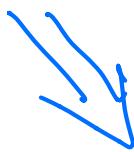
## Differentiation of Inverse Hyperbolic Functions

$$y = \operatorname{arcosh} x = \cosh^{-1} x$$

$$\cosh y = x$$

$$\sinh y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\sinh y}$$



$$\cosh^2 y - \sinh^2 y = 1$$

$$\cosh^2 y = 1 + \sinh^2 y$$

$$\begin{aligned}\cosh^2 y - 1 &= \sinh^2 y \\ \sqrt{\cosh^2 y - 1} &= \sinh y\end{aligned}$$

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{\pm \sqrt{\cosh^2 y - 1}} \\ &= \frac{1}{\pm \sqrt{x^2 - 1}}\end{aligned}$$

Since gradient of  $\text{arccosh } x > 0$

$$\frac{dy}{dx} = \frac{1}{\sqrt{x^2 - 1}}$$

Implication  $y = \cosh^{-1} x \quad \frac{dy}{dx} = \frac{1}{\sqrt{x^2 - 1}}$

$$\Rightarrow \int \frac{1}{\sqrt{x^2 - 1}} dx = \cosh^{-1} x + C$$

This is one example of

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \text{arcosh} \frac{x}{a} + C$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a} + C$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \text{arsinh} \frac{x}{a} + C$$

$$\int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}$$

Ex 1

$$\int \frac{1}{\sqrt{9x^2 - 25}} dx = \int \frac{1}{\sqrt{(3x)^2 - 5^2}} dx$$

$$= \frac{1}{3} \operatorname{arccosh} \frac{3x}{5} + C$$

Let  $u = 3x$

$$\frac{du}{dx} = 3$$

$$du = 3dx$$

$$\frac{1}{3} du = dx$$

$$\int \frac{1}{\sqrt{u^2 - 5^2}} \frac{1}{3} du$$

$$= \frac{1}{3} \operatorname{arccosh} \frac{u}{5} + C$$

$$= \frac{1}{3} \operatorname{arccosh} \frac{3x}{5} + C$$

Ex 2

$$\int_1^5 \frac{1}{\sqrt{x^2 + 6x + 13}} dx$$

$$= \int_1^5 \frac{1}{\sqrt{(x+3)^2 + 2^2}} dx = \left[ \operatorname{arsinh} \left( \frac{x+3}{2} \right) \right]_1^5$$

$$= \operatorname{arsinh} 4 - \operatorname{arsinh} 2$$

$$= 0.651$$

# From Page 21 of Formula Booklet

**Integration (+ constant;  $a > 0$  where relevant)**

$$f(x) \quad \int f(x) dx$$

$$\sinh x \quad \cosh x$$

$$\cosh x \quad \sinh x$$

$$\tanh x \quad \ln \cosh x$$

$$\frac{1}{\sqrt{a^2 - x^2}} \quad \arcsin \left( \frac{x}{a} \right) \quad (|x| < a)$$

$$\frac{1}{a^2 + x^2} \quad \frac{1}{a} \arctan \left( \frac{x}{a} \right)$$

$$\frac{1}{\sqrt{x^2 - a^2}} \quad \operatorname{arcosh} \left( \frac{x}{a} \right), \quad \ln \{x + \sqrt{x^2 - a^2}\} \quad (x > a)$$

$$\frac{1}{\sqrt{a^2 + x^2}} \quad \operatorname{arsinh} \left( \frac{x}{a} \right), \quad \ln \{x + \sqrt{x^2 + a^2}\}$$

$$\frac{1}{a^2 - x^2} \quad \frac{1}{2a} \ln \left| \frac{a+x}{a-x} \right| = \frac{1}{a} \operatorname{artanh} \left( \frac{x}{a} \right) \quad (|x| < a)$$

$$\frac{1}{x^2 - a^2} \quad \frac{1}{2a} \ln \left| \frac{x-a}{x+a} \right|$$

6 Integrate the following with respect to  $x$ .

(i)  $\frac{1}{\sqrt{4 + x^2}}$

(ii)  $\frac{1}{\sqrt{x^2 - 9}}$

(iii)  $\frac{1}{\sqrt{9 - x^2}}$

(iv)  $\frac{1}{\sqrt{36x^2 + 16}}$

(v)  $\frac{1}{\sqrt{x^2 - 4x + 8}}$

(vi)  $\frac{1}{\sqrt{x^2 + x}}$

(vii)  $\frac{1}{\sqrt{9x^2 + 6x - 8}}$

(viii)  $\frac{x^2}{\sqrt{x^6 - 1}}$

7 Evaluate each of the following, correct to 3 significant figures.

(i)  $\int_1^3 \frac{1}{\sqrt{x^2 + 4x + 5}} dx$

(ii)  $\int_{10}^{20} \frac{1}{\sqrt{4x^2 + 12x - 40}} dx$

For Q7 work out using inverse hyperbolic or inverse trig functions as appropriate, then check your answers using integral function on your calculator. Q7 unlikely on exam as definite integrations since too easy on calculator