The determinant of a square matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

written as $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ is given by $ad - bc$

Examples $\underline{M} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

$$\det \underline{M} = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 1 \times 4 - 2 \times 3 = -2$$

$$\underline{P} = \begin{pmatrix} 3 & 4 \\ 6 & 8 \end{pmatrix} \qquad \det \underline{P} = 3 \times 8 - 4 \times 6 = 0$$

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3 x 3 Matrices

Let $\underline{M} = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$

then $\det \underline{M} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$

Example $\begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 3 & 4 & 0 \end{vmatrix}$

$$= 1\begin{vmatrix} 1 & 2 \\ 4 & 0 \end{vmatrix} - 2\begin{vmatrix} 0 & 2 \\ 3 & 0 \end{vmatrix} + 3\begin{vmatrix} 0 & 1 \\ 3 & 4 \end{vmatrix}$$

$$= -8 + 12 - 9$$

$$= -5$$

Inverse of a Matrix

A square matrix $\underline{M}$ has an inverse $\underline{M}^{-1}$ if and only if $\det \underline{M} \neq 0$, and we say it is non-singular

then $\underline{M}\,\underline{M}^{-1} = \underline{M}^{-1}\underline{M} = \underline{I}_n$

$$\underline{I}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \underline{I}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Examples $\quad \underline{M} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \qquad$ Find $\underline{M}^{-1}$

Aside $\quad \underline{M} = \begin{pmatrix} a & d \\ c & b \end{pmatrix}$

$$M^{-1} = \frac{1}{\det M}\begin{pmatrix} b & -d \\ -c & a \end{pmatrix} = \frac{1}{ab-cd}\begin{pmatrix} b & -d \\ -c & a \end{pmatrix}$$

if $\det \underline{M} \neq 0$

$$\underline{M}^{-1} = \frac{1}{-2}\begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} -2+3 & 1-1 \\ -6+6 & 3-2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Simultaneous Equations

$$2x + 3y = 14$$
$$5x - 2y = 16$$

$$\begin{pmatrix} 2 & 3 \\ 5 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 14 \\ 16 \end{pmatrix}$$

Let $\underline{M} = \begin{pmatrix} 2 & 3 \\ 5 & -2 \end{pmatrix}$

then $\underline{M}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 14 \\ 16 \end{pmatrix}$

$$\underline{M}^{-1}\underline{M}\begin{pmatrix} x \\ y \end{pmatrix} = \underline{M}^{-1}\begin{pmatrix} 14 \\ 16 \end{pmatrix}$$

$$\underline{I}\begin{pmatrix} x \\ y \end{pmatrix} = \underline{M}^{-1}\begin{pmatrix} 14 \\ 16 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \underline{M}^{-1}\begin{pmatrix} 14 \\ 16 \end{pmatrix}$$

$$\underline{M}^{-1} = \frac{1}{-19}\begin{pmatrix} -2 & -3 \\ -5 & 2 \end{pmatrix} = \frac{1}{19}\begin{pmatrix} 2 & 3 \\ 5 & -2 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{19}\begin{pmatrix} 2 & 3 \\ 5 & -2 \end{pmatrix}\begin{pmatrix} 14 \\ 16 \end{pmatrix} = \frac{1}{19}\begin{pmatrix} 76 \\ 38 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$$

Ex 2 Solve

$$\begin{aligned} -x + 6y - 2z &= 21 \\ 6x - 2y - z &= 16 \\ -2x + 3y + 5z &= 24 \end{aligned}$$

$$\begin{pmatrix} -1 & 6 & -2 \\ 6 & -2 & -1 \\ -2 & 3 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 21 \\ -16 \\ 24 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 & 6 & -2 \\ 6 & -2 & -1 \\ -2 & 3 & 5 \end{pmatrix}^{-1}\begin{pmatrix} 21 \\ -16 \\ 24 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \\ 2 \end{pmatrix}$$

Ex 6F Q4 Brown, Grey, Red

$$\begin{aligned} B + G + R &= 2000 \\ B - G &= 250 \\ .99B + 0.98G + 1.04R &= 2040 \end{aligned}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ .99 & .98 & 1.04 \end{pmatrix} \begin{pmatrix} B \\ G \\ R \end{pmatrix} = \begin{pmatrix} 2000 \\ 250 \\ 2040 \end{pmatrix}$$

$$\begin{pmatrix} B \\ G \\ R \end{pmatrix} = \begin{pmatrix} \quad \end{pmatrix}^{-1} \begin{pmatrix} 2000 \\ 250 \\ 2040 \end{pmatrix} = \begin{pmatrix} 500 \\ 250 \\ 1250 \end{pmatrix}$$
