General Motion - MEI 2005-2009

Jan 05

1 The position vector, **r**, of a particle of mass 4 kg at time t is given by

$$\mathbf{r} = t^2\mathbf{i} + (5t - 2t^2)\mathbf{j},$$

where **i** and **j** are the standard unit vectors, lengths are in metres and time is in seconds.

(i) Find an expression for the acceleration of the particle. [4]

The particle is subject to a force **F** and a force $12\mathbf{j}$ N.

(ii) Find **F**. [3]

i)

$$\underline{r} = \begin{pmatrix} t^2 \\ 5t - 2t^2 \end{pmatrix}$$

$$\underline{v} = \frac{d\underline{r}}{dt} = \begin{pmatrix} 2t \\ 5 - 4t \end{pmatrix}$$

$$\underline{a} = \frac{d\underline{v}}{dt} = \begin{pmatrix} 2 \\ -4 \end{pmatrix} = 2\underline{i} - 4\underline{j} \quad \text{m s}^{-2}$$

ii)

$$\underline{F} + \begin{pmatrix} 0 \\ 12 \end{pmatrix} = m\underline{a}$$

$$\underline{F} + \begin{pmatrix} 0 \\ 12 \end{pmatrix} = 4\begin{pmatrix} 2 \\ -4 \end{pmatrix}$$

$$\underline{F} = \begin{pmatrix} 8 \\ -16 \end{pmatrix} - \begin{pmatrix} 0 \\ 12 \end{pmatrix}$$

$$\underline{F} = \begin{pmatrix} 8 \\ -28 \end{pmatrix} = 8\underline{i} - 28\underline{j} \quad \text{N}$$

Jun 05

5 The position vector of a particle at time t is given by

$$\mathbf{r} = \tfrac{1}{2}t\mathbf{i} + (t^2 - 1)\mathbf{j},$$

referred to an origin O where **i** and **j** are the standard unit vectors in the directions of the cartesian axes Ox and Oy respectively.

(i) Write down the value of t for which the x-coordinate of the position of the particle is 2. Find the y-coordinate at this time. [2]

(ii) Show that the cartesian equation of the path of the particle is $y = 4x^2 - 1$. [2]

(iii) Find the coordinates of the point where the particle is moving at 45° to both Ox and Oy. [3]

i) $x = 2$ when $\frac{1}{2}t = 2 \Rightarrow \underline{t = 4}$

when $t = 4$, $y = (4^2 - 1) = 15$ $\quad \underline{y = 15}$

ii) $\underline{r} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{2}t \\ t^2 - 1 \end{pmatrix}$

$x = \frac{1}{2}t \Rightarrow t = 2x$

Sub for t in $y = t^2 - 1$

$y = (2x)^2 - 1$ $\quad \underline{y = 4x^2 - 1}$

iii) $\underline{v} = \frac{d\underline{r}}{dt} = \begin{pmatrix} \frac{1}{2} \\ 2t \end{pmatrix}$

Travelling at 45° to Ox, Oy when velocity components are equal

$\Rightarrow \frac{1}{2} = 2t$

$\Rightarrow \frac{1}{4} = t$

$\underline{t = \frac{1}{4}}$

When $t = \frac{1}{4}$ $\quad \underline{r} = \begin{pmatrix} \frac{1}{2}\left(\frac{1}{4}\right) \\ \left(\frac{1}{4}\right)^2 - 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{8} \\ -\frac{15}{16} \end{pmatrix}$

Point is $\left(\frac{1}{8}, -\frac{15}{16}\right)$

Jan 06

5 The acceleration of a particle of mass 4 kg is given by $\mathbf{a} = (9\mathbf{i} - 4t\mathbf{j})\ \mathrm{m\,s^{-2}}$, where $\mathbf{i}$ and $\mathbf{j}$ are unit vectors and t is the time in seconds.

(i) Find the acceleration of the particle when $t = 0$ and also when $t = 3$. [1]

(ii) Calculate the force acting on the particle when $t = 3$. [1]

The particle has velocity $(4\mathbf{i} + 2\mathbf{j})\ \mathrm{m\,s^{-1}}$ when $t = 1$.

(iii) Find an expression for the velocity of the particle at time t. [4]

i) $t = 0, \quad \underline{a} = 9\underline{i} \ \mathrm{ms^{-2}}$

$t = 3, \quad \underline{a} = 9\underline{i} - 12\underline{j} \ \mathrm{ms^{-2}}$

ii) $\underline{F} = m\underline{a} = 4(9\underline{i} - 12\underline{j}) = 36\underline{i} - 48\underline{j}$ N

iii) $\underline{v} = \int \underline{a}\, dt = \int \begin{pmatrix} 9 \\ -4t \end{pmatrix} dt$

$\underline{v} = \begin{pmatrix} 9t \\ -2t^2 \end{pmatrix} + \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$

$t = 1, \quad \underline{v} = \begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 9(1) \\ -2(1)^2 \end{pmatrix} + \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$

$$\begin{pmatrix}4\\2\end{pmatrix} - \begin{pmatrix}9\\-2\end{pmatrix} = \begin{pmatrix}c_1\\c_2\end{pmatrix}$$

$$\begin{pmatrix}-5\\4\end{pmatrix} = \begin{pmatrix}c_1\\c_2\end{pmatrix}$$

$$\underline{v} = \begin{pmatrix}9t-5\\-2t^2+4\end{pmatrix}$$

$$\underline{v} = (9t-5)\underline{i} + (-2t^2+4)\underline{j}$$

Jun 06

4 Fig. 4 shows the unit vectors **i** and **j** in the directions of the cartesian axes Ox and Oy, respectively. O is the origin of the axes and of position vectors.

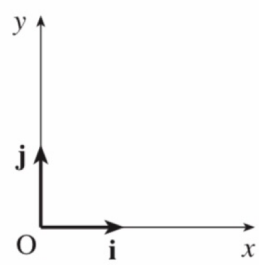


Fig. 4

The position vector of a particle is given by $\mathbf{r} = 3t\mathbf{i} + (18t^2 - 1)\mathbf{j}$ for $t \geqslant 0$, where t is time.

(i) Show that the path of the particle cuts the x-axis just once. [2]

(ii) Find an expression for the velocity of the particle at time t.

Deduce that the particle never travels in the **j** direction. [3]

(iii) Find the cartesian equation of the path of the particle, simplifying your answer. [3]

i) Cuts x-axis when $18t^2 - 1 = 0$

$$18t^2 = 1$$

$$t^2 = \frac{1}{18}$$

$t = \frac{1}{\sqrt{18}}$ or $\cancel{-\frac{1}{\sqrt{18}}}$ since $t \geq 0$

$\therefore$ only once at $t = \frac{1}{\sqrt{18}}$ s

---

ii) $\underline{r} = \begin{pmatrix} 3t \\ 18t^2 - 1 \end{pmatrix}$

$\underline{v} = \frac{d\underline{r}}{dt} = \begin{pmatrix} 3 \\ 36t \end{pmatrix} = 3\underline{i} + 36t\underline{j}$ ms^{-1}

Never travels in $\underline{j}$ direction as velocity always has $\underline{i}$ component of 3

---

iii) $\underline{r} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3t \\ 18t^2 - 1 \end{pmatrix}$

$x = 3t \Rightarrow t = \frac{x}{3}$

Sub for t $\quad y = 18\left(\frac{x}{3}\right)^2 - 1$

$y = 2x^2 - 1$

Jan 07

6 The velocity of a model boat, $\mathbf{v}\,\mathrm{m\,s^{-1}}$, is given by

$$\mathbf{v} = \begin{pmatrix} -5 \\ 10 \end{pmatrix} + t\begin{pmatrix} 6 \\ -8 \end{pmatrix},$$

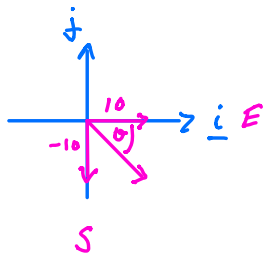
where t is the time in seconds and the vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are east and north respectively.

(i) Show that when $t = 2.5$ the boat is travelling south-east (i.e. on a bearing of 135°). Calculate its speed at this time. [3]

The boat is at a point O when $t = 0$.

(ii) Calculate the bearing of the boat from O when $t = 2.5$. [4]

i) $t = 2.5$ $\quad \underline{v} = \begin{pmatrix} -5 \\ 10 \end{pmatrix} + 2.5\begin{pmatrix} 6 \\ -8 \end{pmatrix} = \begin{pmatrix} 10 \\ -10 \end{pmatrix}\ \mathrm{m\,s^{-1}}$



$\theta = \tan^{-1}\left(\frac{10}{10}\right) = 45°$

Travelling South East

speed $= \sqrt{10^2 + (-10)^2}$

$= 10\sqrt{2}\ \mathrm{m\,s^{-1}}$

$= 14.1\ \mathrm{m\,s^{-1}}$

ii) $\underline{r} = \int \underline{v}\,dt = \int \begin{pmatrix} -5 + 6t \\ 10 - 8t \end{pmatrix} dt$

$\underline{r} = \begin{pmatrix} -5t + 3t^2 + c_1 \\ 10t - 4t^2 + c_2 \end{pmatrix}$

$t = 0,\ \underline{r} = 0 \Rightarrow c_1, c_2 = 0$

$\therefore\ \underline{r} = \begin{pmatrix} -5t + 3t^2 \\ 10t - 4t^2 \end{pmatrix}$

$t = 2.5, \quad \underline{r} = \begin{pmatrix} -5(2.5) + 3(2.5)^2 \\ 10(2.5) - 4(2.5)^2 \end{pmatrix} = \begin{pmatrix} 6.25 \\ 0 \end{pmatrix}$

$\therefore$ position is East of O

Bearing 090°

Jun 07

6 A rock of mass 8 kg is acted on by just the two forces $-80\mathbf{k}$ N and $(-\mathbf{i} + 16\mathbf{j} + 72\mathbf{k})$ N, where $\mathbf{i}$ and $\mathbf{j}$ are perpendicular unit vectors in a horizontal plane and $\mathbf{k}$ is a unit vector vertically upward.

(i) Show that the acceleration of the rock is $\left(-\frac{1}{8}\mathbf{i} + 2\mathbf{j} - \mathbf{k}\right)\text{m s}^{-2}$. [2]

The rock passes through the origin of position vectors, O, with velocity $(\mathbf{i} - 4\mathbf{j} + 3\mathbf{k})$ m s^{-1} and 4 seconds later passes through the point A.

(ii) Find the position vector of A. [3]

(iii) Find the distance OA. [1]

(iv) Find the angle that OA makes with the horizontal. [2]

i) $\underline{F} = m\underline{a}$

$$\begin{pmatrix} 0 \\ 0 \\ -80 \end{pmatrix} + \begin{pmatrix} -1 \\ 16 \\ 72 \end{pmatrix} = 8\underline{a}$$

$$\begin{pmatrix} -1 \\ 16 \\ -8 \end{pmatrix} = 8\underline{a}$$

$$\underline{a} = \frac{1}{8}\begin{pmatrix} -1 \\ 16 \\ -8 \end{pmatrix} = \begin{pmatrix} -\frac{1}{8} \\ 2 \\ -1 \end{pmatrix}$$

$$\underline{a} = \left(-\frac{1}{8}\underline{i} + 2\underline{j} - \underline{k}\right) \text{ m s}^{-2}$$

Constant acceleration so can use SUVAT

ii) At O, $t = 0$, $\underline{u} = \begin{pmatrix} 1 \\ -4 \\ 3 \end{pmatrix}$

$$\underline{r} = \underline{u}t + \tfrac{1}{2}\underline{a}t^2$$

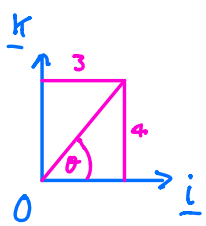
$t = 4$, $$\underline{r} = 4\begin{pmatrix} 1 \\ -4 \\ 3 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} -\frac{1}{8} \\ 2 \\ -1 \end{pmatrix} \times 4^2$$

$$\underline{r} = \begin{pmatrix} 4 \\ -16 \\ 12 \end{pmatrix} + \begin{pmatrix} -1 \\ 16 \\ -8 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix}$$

$$\underline{\overrightarrow{OA} = 3\underline{i} + 4\underline{k}}$$

iii) $$|\overrightarrow{OA}| = \sqrt{3^2 + 4^2} = 5 \text{ m}$$

iv)



$$\theta = \tan^{-1}\left(\frac{4}{3}\right)$$

$$\underline{\theta = 53.1^\circ}$$

Jan 08

2 The force acting on a particle of mass 1.5 kg is given by the vector $\begin{pmatrix} 6 \\ 9 \end{pmatrix}$ N.

(i) Give the acceleration of the particle as a vector. [2]

(ii) Calculate the angle that the acceleration makes with the direction $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. [2]

(iii) At a certain point of its motion, the particle has a velocity of $\begin{pmatrix} -2 \\ 3 \end{pmatrix}$ m s^{-1}. Calculate the displacement of the particle over the next two seconds. [3]

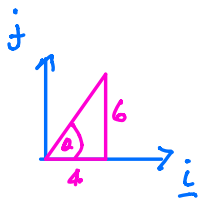
i) $\underline{F} = m\underline{a}$

$$\begin{pmatrix} 6 \\ 9 \end{pmatrix} = 1.5\,\underline{a}$$

$$\underline{a} = \frac{1}{1.5}\begin{pmatrix} 6 \\ 9 \end{pmatrix}$$

$$\underline{a} = \begin{pmatrix} 4 \\ 6 \end{pmatrix} \text{ ms}^{-2}$$

ii)



$$\theta = \tan^{-1}\left(\frac{6}{4}\right)$$

$$\theta = 56.3^\circ$$

iii) Start clock $t = 0$, $\underline{r} = 0$, when $\underline{v} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$ ms^{-1}

$$\underline{r} = \underline{v}t + \tfrac{1}{2}\underline{a}t^2$$

$$\underline{r} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}t + \tfrac{1}{2}\begin{pmatrix} 4 \\ 6 \end{pmatrix}t^2$$

$t = 2$

$$\underline{r} = \begin{pmatrix} -2 \\ 3 \end{pmatrix} \times 2 + \tfrac{1}{2}\begin{pmatrix} 4 \\ 6 \end{pmatrix} \times 2^2$$

$$\underline{r} = \begin{pmatrix} -4 \\ 6 \end{pmatrix} + \begin{pmatrix} 8 \\ 12 \end{pmatrix} = \begin{pmatrix} 4 \\ 18 \end{pmatrix}$$

Over next 2 seconds

$$\underline{r} = \begin{pmatrix} 4 \\ 18 \end{pmatrix} \text{ m}$$

Jun 08

3 An object of mass 5 kg has a constant acceleration of $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ m s^{-2} for $0 \leqslant t \leqslant 4$, where t is the time in seconds.

(i) Calculate the force acting on the object. [2]

When $t = 0$, the object has position vector $\begin{pmatrix} -2 \\ 3 \end{pmatrix}$ m and velocity $\begin{pmatrix} 4 \\ 5 \end{pmatrix}$ m s^{-1}.

(ii) Find the position vector of the object when $t = 4$. [3]

i) $\underline{F} = m\underline{a}$ $\qquad \underline{F} = 5\begin{pmatrix} -1 \\ 2 \end{pmatrix}$

$\underline{F} = \begin{pmatrix} -5 \\ 10 \end{pmatrix}$ N

---

ii) $t = 0$, $\underline{r}_0 = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$, $\underline{u} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$

$\underline{r} - \underline{r}_0 = \underline{u}t + \frac{1}{2}\underline{a}t^2$

$t = 4$, $\quad \underline{r} - \begin{pmatrix} -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \end{pmatrix} \times 4 + \frac{1}{2}\begin{pmatrix} -1 \\ 2 \end{pmatrix} \times 4^2$

$\underline{r} = \begin{pmatrix} 16 \\ 20 \end{pmatrix} + \begin{pmatrix} -8 \\ 16 \end{pmatrix} + \begin{pmatrix} -2 \\ 3 \end{pmatrix}$

$\underline{r} = \begin{pmatrix} 6 \\ 39 \end{pmatrix}$ m

Jan 09

8 A toy boat moves in a horizontal plane with position vector $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$, where $\mathbf{i}$ and $\mathbf{j}$ are the standard unit vectors east and north respectively. The origin of the position vectors is at O. The displacements x and y are in metres.

First consider only the motion of the boat parallel to the x-axis. For this motion

$$x = 8t - 2t^2.$$

The velocity of the boat in the x-direction is $v_x \text{ m s}^{-1}$.

(i) Find an expression in terms of t for v_x and determine when the boat instantaneously has zero speed in the x-direction. **[3]**

Now consider only the motion of the boat parallel to the y-axis. For this motion

$$v_y = (t - 2)(3t - 2),$$

where $v_y \text{ m s}^{-1}$ is the velocity of the boat in the y-direction at time t seconds.

(ii) Given that $y = 3$ when $t = 1$, use integration to show that $y = t^3 - 4t^2 + 4t + 2$. **[4]**

i) $V_x = \frac{dx}{dt}$ $\quad x = 8t - 2t^2$

$V_x = 8 - 4t$

$V_x = 0 \Rightarrow 8 - 4t = 0 \Rightarrow t = 2 \text{ s}$

ii) $V_y = (t-2)(3t-2) = 3t^2 - 8t + 4$

$y = \int V_y \, dt$ $\quad y = t^3 - 4t^2 + 4t + c$

$y = 3, t = 1$ $\quad 3 = 1^3 - 4(1)^2 + 4(1) + c$

$3 = 1 + c$

$2 = c$

$y = t^3 - 4t^2 + 4t + 2$

The position vector of the boat is given in terms of t by $\mathbf{r} = (8t - 2t^2)\mathbf{i} + (t^3 - 4t^2 + 4t + 2)\mathbf{j}$.

(iii) Find the time(s) when the boat is due north of O and also the distance of the boat from O at any such times. **[4]**

(iv) Find the time(s) when the boat is instantaneously at rest. Find the distance of the boat from O at any such times. **[5]**

(v) Plot a graph of the path of the boat for $0 \leqslant t \leqslant 2$. **[3]**

iii) $\underline{r} = \begin{pmatrix} 8t - 2t^2 \\ t^3 - 4t^2 + 4t + 2 \end{pmatrix}$

Due North of O when $8t - 2t^2 = 0$

$2t(4 - t) = 0$

$\Rightarrow t = 0$ or $t = 4$ s

When $t = 0$, $\underline{r} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$

so distance from O $= 2$ m

When $t = 4$ $\underline{r} = \begin{pmatrix} 8(4) - 2(4)^2 \\ 4^3 - 4(4)^2 + 4(4) + 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 18 \end{pmatrix}$

so distance from O $= 16$ m

iv) At rest when $V_x = 0$ and $V_y = 0$

$V_x = 0$ when $t = 2$, $V_y = 0$ when $t = 2$, or $t = \frac{2}{3}$

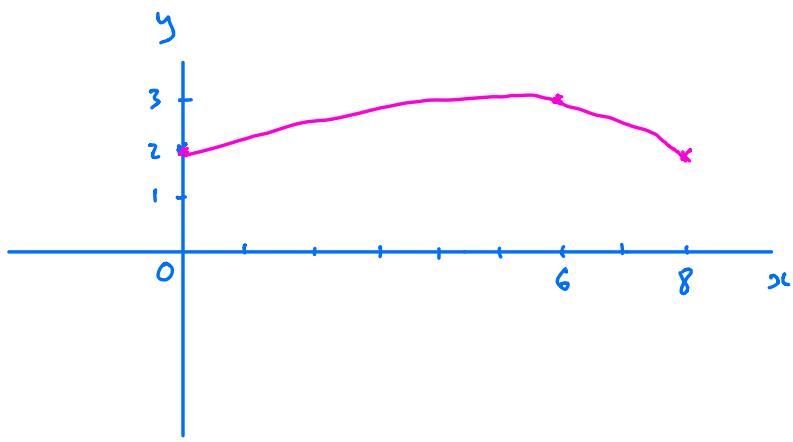
$\therefore$ at rest only when $t = 2$ s

$t = 2,\quad \underline{r} = \begin{pmatrix} 8(2) - 2(2)^2 \\ 2^3 - 4(2)^2 + 4(2) + 2 \end{pmatrix} = \begin{pmatrix} 8 \\ 2 \end{pmatrix}$

Distance from O $= \sqrt{8^2 + 2^2} = \sqrt{68}$ m

$= 8.25$ m

v)



$t = 0\ (0, 2)$

$t = 1\ (6, 3)$

$t = 2\ (8, 2)$

Jun 09

5 The position vector of a toy boat of mass 1.5 kg is modelled as $\mathbf{r} = (2 + t)\mathbf{i} + (3t - t^2)\mathbf{j}$ where lengths are in metres, t is the time in seconds, **i** and **j** are horizontal, perpendicular unit vectors and the origin is O.

(i) Find the velocity of the boat when $t = 4$. **[3]**

(ii) Find the acceleration of the boat and the horizontal force acting on the boat. **[3]**

(iii) Find the cartesian equation of the path of the boat referred to x- and y-axes in the directions of **i** and **j**, respectively, with origin O. You are not required to simplify your answer. **[2]**

i) $\underline{r} = \begin{pmatrix} 2 + t \\ 3t - t^2 \end{pmatrix} \qquad \underline{v} = \dfrac{d\underline{r}}{dt} = \begin{pmatrix} 1 \\ 3 - 2t \end{pmatrix}$

when $t = 4$, $\underline{v} = \begin{pmatrix} 1 \\ 3 - 2(4) \end{pmatrix} = \begin{pmatrix} 1 \\ -5 \end{pmatrix} = \underline{i} - 5\underline{j}$ m s^{-1}

ii) $\underline{a} = \dfrac{d\underline{v}}{dt} = \begin{pmatrix} 0 \\ -2 \end{pmatrix} = \underline{-2\underline{j} \ \text{ms}^{-2}}$

$\underline{F} = m\underline{a} \qquad \underline{F} = 1.5\begin{pmatrix} 0 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ -3 \end{pmatrix}$

$\underline{\underline{F = -3\underline{j} \ \text{N}}}$

iii) $\underline{r} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2+t \\ 3t - t^2 \end{pmatrix}$

$x = 2 + t \Rightarrow t = x - 2$

Sub for t $\quad \underline{\underline{y = 3(x-2) - (x-2)^2}}$