Logs and Exponentials

1 The graph of $y = ab^x$ passes through the points (2, 400) and (5, 50).

a Find the values of the constants a and b. **(5 marks)**

b Given that $ab^x < k$, for some constant $k > 0$, show that $x > \dfrac{\log\left(\dfrac{1600}{k}\right)}{\log 2}$ where log means log to any valid base. **(4 marks)**

a)

$$400 = ab^2 \quad (1)$$
$$50 = ab^5 \quad (2)$$

$(2) \div (1)$

$$\frac{50}{400} = \frac{ab^5}{ab^2}$$

$$\frac{1}{8} = b^3 \quad \Rightarrow \quad \underline{b = \sqrt{\tfrac{1}{8}} = \tfrac{1}{2}}$$

Sub in (1)

$$400 = a\left(\tfrac{1}{2}\right)^2$$

$$400 = \frac{a}{4} \quad \Rightarrow \quad \underline{a = 1600}$$

b)

$$ab^x < k$$

$$1600 \times \left(\tfrac{1}{2}\right)^x < k$$

$$\log\left(1600 \times \left(\tfrac{1}{2}\right)^x\right) < \log k$$

$$\log 1600 + \log\left(\tfrac{1}{2}\right)^x < \log k$$

$$\log 1600 + x \log \tfrac{1}{2} < \log k$$

$$\log 1600 + x \log 2^{-1} < \log k$$

$$\log 1600 - x \log 2 < \log k$$

$$\log 1600 - \log k < x \log 2$$

$$\frac{\log\left(\frac{1600}{k}\right)}{\log 2} < x$$

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2 $\log_{11}(2x-1) = 1 - \log_{11}(x+4)$

Find the value of x showing detailed reasoning. **(6 marks)**

$$\log_{11}(2x-1) + \log_{11}(x+4) = 1$$

$$\log_{11}\big((2x-1)(x+4)\big) = 1$$

$$(2x-1)(x+4) = 11^1$$

$$2x^2 - x + 8x - 4 - 11 = 0$$

$$2x^2 + 7x - 15 = 0$$

$$(2x-3)(x+5) = 0$$

$$\Rightarrow \quad \underline{x = \tfrac{3}{2}} \quad \text{or} \quad \cancel{x = -5}$$

leads to log of -ve number

6 The population, P, of bacteria in an experiement can be modelled by the formula $P = 100e^{0.4t}$, where t is the time in hours after the experiment began.

a Use the model to estimate the population of bacteria 7 hours after the experiment began. **(2 marks)**

b Interpret the meaning of the constant 100 in the model. **(1 mark)**

c How many whole hours after the experiment began does the population of bacteria first exceed 1 million, according to the model? **(3 marks)**

a) $P = 100e^{0.4 \times 7} = 1644.46$

$= \underline{1644}$

b) 100 is initial population when $t = 0$

c) $100e^{0.4t} > 1000000$

$e^{0.4t} > \frac{1000000}{100}$

$0.4t > \ln(10000)$

$t > \frac{\ln(10000)}{0.4}$

$t > 23.03$ years

$\underline{t = 24 \text{ years}}$

7 The value of a car, V in £, is modelled by the equation $V = ab^t$, where a and b are constants and t is the number of years since the car was purchased. The line l shown in Figure 1 illustrates the linear relationship between t and $\log_4 V$ for t Ö 0. The line l meets the vertical axis at

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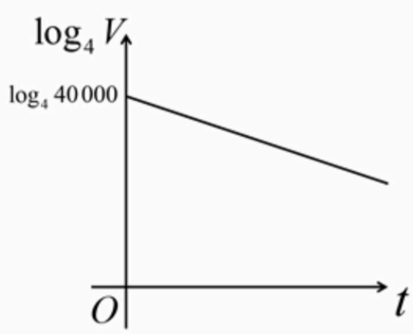
(0, $\log_4 40\,000$) as shown. The gradient of l is $-\frac{1}{10}$.

Figure 1



a Write down an equation for l. **(2 marks)**

b Find, in exact form, the values of a and b. **(4 marks)**

c With reference to the model, interpret the values of the constant a and b. **(2 marks)**

d Find the value of the car after 7 years. **(1 mark)**

e After how many years is the value of the car less than £10 000? **(2 marks)**

f State a limitation of the model. **(1 mark)**

a) $\log_4 V = -\frac{1}{10}t + \log_4 40000$

b) $V = ab^t$

$\log_4 V = \log_4 (ab^t)$

$\log_4 V = \log_4 a + \log_4 b^t$

$\log_4 V = \log_4 a + t \log_4 b$

$\Rightarrow \quad a = 40000 \qquad \log_4 b = -\frac{1}{10}$

$b = 4^{-\frac{1}{10}}$

c) $a = 40000$ is the initial value when $t = 0$

b is the proportional multiplier each year

$b \approx 0.87$ showing depreciation of 13 % per annum

d) $V = 40000 \times \left(4^{-\frac{1}{10}}\right)^t$

$V = 40000 \times 4^{-0.1t}$

After 7 years $\quad V = 40000 \times 4^{-0.1 \times 7} = £15157$

e) $40000 \times 4^{-0.1t} < 10000$

$4^{-0.1t} < \frac{10000}{40000}$

$4^{-0.1t} < \frac{1}{4}$

$$-0.1t\ln 4 < \ln \tfrac{1}{4}$$

$$t > \frac{\ln \frac{1}{4}}{(-0.1\ln 4)}$$

$$\underline{t > 10 \text{ years}}$$

f) Car never becomes worthless even in very long term

Value dependent on other factors not just age. eg mileage.