Stationary Points and Modelling Questions (52 Marks)

Q1.

The curve with equation

$$y = x^2 - 32\sqrt{(x)} + 20, \quad x > 0$$

has a stationary point P.

Use calculus

(a) to find the coordinates of P,

(6)

(b) to determine the nature of the stationary point P.

(3)

(Total 9 marks)

a)

$$y = x^2 - 32x^{\frac{1}{2}} + 20$$

$$\frac{dy}{dx} = 2x - 16x^{-\frac{1}{2}} = 2x - \frac{16}{\sqrt{x}}$$

At st. pt. $\frac{dy}{dx} = 0 \Rightarrow 2x - \frac{16}{\sqrt{x}} = 0$

$$2x^{3/2} - 16 = 0$$

$$2x^{3/2} = 16$$

$$x^{3/2} = 8$$

$$x = 8^{2/3}$$

$$\underline{x = 4}$$

$$y = 4^2 - 32\sqrt{4} + 20$$

$$y = 16 - 64 + 20$$

$$y = -28$$

$\therefore \quad P(4, -28)$

b)

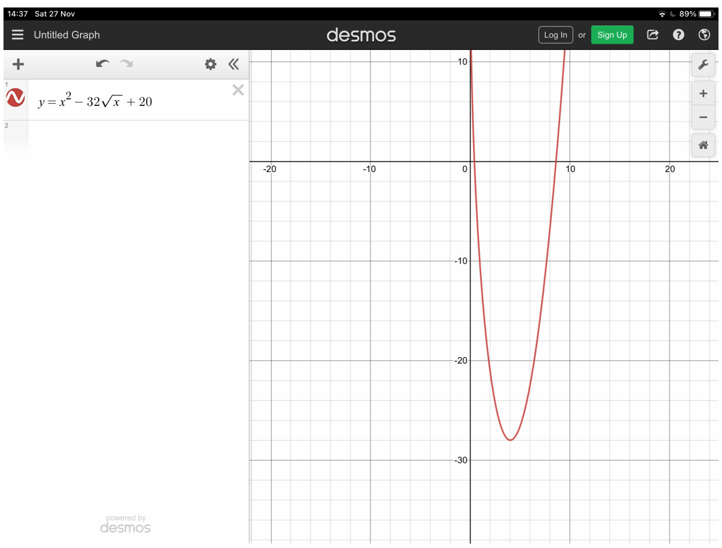
$$\frac{dy}{dx} = 2x - 16x^{-\frac{1}{2}}$$

$$\frac{d^2y}{dx^2} = 2 + 8x^{-3/2} = 2 + \frac{8}{(\sqrt{x})^3}$$

when $x = 4$, $$\frac{d^2y}{dx^2} = 2 + \frac{8}{(\sqrt{4})^3}$$

$$= 2 + 1$$

$$= 3 > 0 \therefore \text{ a minimum}$$



Q2.

The curve C has equation $y = 12\sqrt{(x)} - x^{\frac{3}{2}} - 10$, $x > 0$

(a) Use calculus to find the coordinates of the turning point on C.

(7)

(b) Find $\frac{d^2y}{dx^2}$.

(2)

(c) State the nature of the turning point.

(1)

(Total 10 marks)

a) $y = 12x^{1/2} - x^{3/2} - 10$

$\frac{dy}{dx} = 6x^{-\frac{1}{2}} - \frac{3}{2}x^{1/2}$

At t.p. $\frac{dy}{dx} = 0 \Rightarrow 6x^{-\frac{1}{2}} - \frac{3}{2}x^{\frac{1}{2}} = 0$

$(\times x^{\frac{1}{2}})$ $6 - \frac{3}{2}x = 0$

$12 - 3x = 0$

$12 = 3x$

$\underline{x = 4}$

$y = 12(4)^{\frac{1}{2}} - (4)^{3/2} - 10$

$y = 24 - 8 - 10$

$\underline{y = 6}$

<u>Turning point is $(4, 6)$</u>

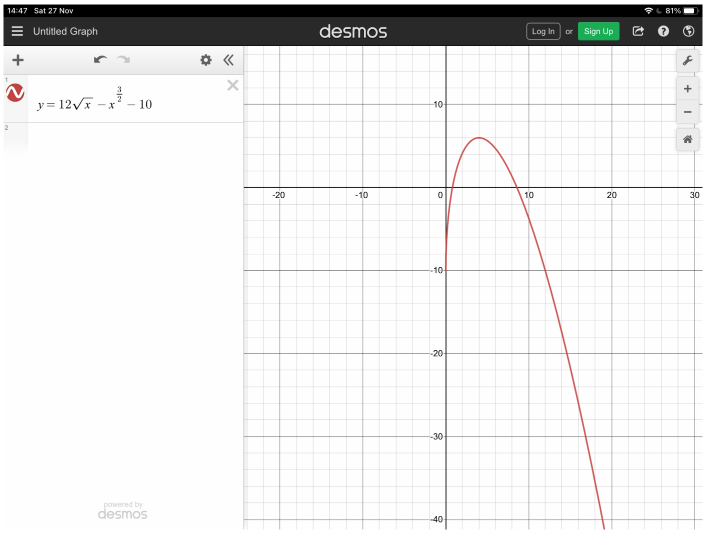
b) $\frac{dy}{dx} = 6x^{-\frac{1}{2}} - \frac{3}{2}x^{\frac{1}{2}}$

$\underline{\frac{d^2y}{dx^2} = -3x^{-3/2} - \frac{3}{4}x^{-\frac{1}{2}}}$

c) When $x = 4$, $\frac{d^2y}{dx^2} = \frac{-3}{4^{3/2}} - \frac{3}{4(4)^{\frac{1}{2}}}$

$= -\frac{3}{8} - \frac{3}{8} = -\frac{6}{8} < 0$

<u>$\therefore$ turning point is a maximum</u>



Q3.

The volume V cm^3 of a box, of height x cm, is given by

$$V = 4x(5 - x)^2, \qquad 0 < x < 5$$

(a) Find $\frac{dV}{dx}$.

(4)

(b) Hence find the maximum volume of the box.

(4)

(c) Use calculus to justify that the volume that you found in part (b) is a maximum.

(2)

(Total 10 marks)

a) $V = 4x(5-x)^2$

$V = 4x(25 - 10x + x^2)$

$V = 100x - 40x^2 + 4x^3$

$\frac{dV}{dx} = 100 - 80x + 12x^2$

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b) Max Vol when $\frac{dV}{dx} = 0$

$\Rightarrow \quad 12x^2 - 80x + 100 = 0$

$$3x^2 - 20x + 25 = 0$$

$$(3x-5)(x-5) = 0$$

$$\Rightarrow \quad x = \frac{5}{3} \quad \text{or} \quad x = 5$$

Volume $= 0$ when $x = 5$

Max volume will be when $x = \frac{5}{3}$

$$V = 4\left(\frac{5}{3}\right)\left(5 - \frac{5}{3}\right)^2$$

$$V = 4 \times \frac{5}{3} \times \frac{10}{3} \times \frac{10}{3}$$

$$\text{max } V = \frac{2000}{27} \text{ cm}^3 = 74.1 \text{ cm}^3 \text{ to } 3 \text{ s.f.}$$

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c)

$$\frac{dV}{dx} = 100 - 80x + 12x^2$$

$$\frac{d^2V}{dx^2} = -80 + 24x$$

When $x = \frac{5}{3}$, $\frac{d^2V}{dx^2} = -80 + 24 \times \frac{5}{3}$

$$= -80 + 40$$

$$= -40 < 0$$

$\therefore$ a maximum

Q4.

A solid right circular cylinder has radius *r* cm and height *h* cm.

The total surface area of the cylinder is 800 cm^2.

(a) Show that the volume, V cm^3, of the cylinder is given by

$$V = 400r - \pi r^3.$$

(4)

Given that *r* varies,

(b) use calculus to find the maximum value of V, to the nearest cm^3.

(6)

(c) Justify that the value of V you have found is a maximum.

(2)

(Total 12 marks)

a) Surface Area $= 2\pi r^2 + 2\pi rh$

$\Rightarrow \quad 800 = 2\pi r^2 + 2\pi rh$

$800 - 2\pi r^2 = 2\pi rh$

$$h = \frac{800 - 2\pi r^2}{2\pi r}$$

$$\text{Volume} = \pi r^2 h = \pi r^2\left(\frac{800 - 2\pi r^2}{2\pi r}\right)$$

$$= r(400 - \pi r^2)$$

$$\underline{\text{Volume} = 400r - \pi r^3}$$

b) $V = 400r - \pi r^3$

$$\frac{dV}{dr} = 400 - 3\pi r^2$$

At max volume $\frac{dV}{dr} = 0$

$$\Rightarrow \quad 400 - 3\pi r^2 = 0$$

$$400 = 3\pi r^2$$

$$\frac{400}{3\pi} = r^2$$

$$r = \sqrt{\frac{400}{3\pi}} = 6.515$$

$$\text{Max Volume} = 400 \times 6.515 - \pi \times 6.515^3$$

$$= 1737 \text{ cm}^3$$

c)

$$\frac{dV}{dr} = 400 - 3\pi r^2$$

$$\frac{d^2V}{d} = -6\pi r$$

r^2

When $r = 6.515$, $\frac{d^2V}{dr^2} = -6\pi \times 6.515$

$$= -122.8 < 0$$

$\therefore$ a maximum

Q5.

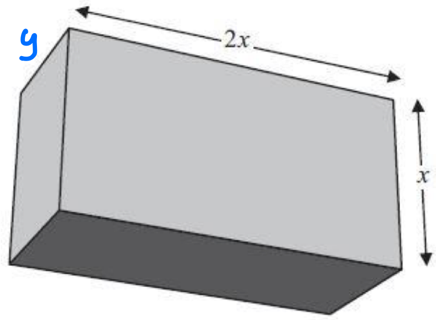

Figure 2

A cuboid has a rectangular cross-section where the length of the rectangle is equal to twice its width, x cm, as shown in Figure 2.
The volume of the cuboid is 81 cubic centimetres.

(a) Show that the total length, L cm, of the twelve edges of the cuboid is given by

$$L = 12x + \frac{162}{x^2}$$

(3)

(b) Use calculus to find the minimum value of L.

(6)

(c) Justify, by further differentiation, that the value of L that you have found is a minimum.

(2)

(Total 11 marks)

a) Let depth be y

Volume $= 2x \times x \times y = 2x^2y$ cm^3

$\Rightarrow 2x^2y = 81$

$y = \frac{81}{2x^2}$

Total length of 12 edges

$L = 4x + 4(2x) + 4y$

$L = 12x + 4\left(\frac{81}{2x^2}\right)$

$L = 12x + \frac{162}{x^2}$

b) $L = 12x + 162x^{-2}$

$$\frac{dL}{dx} = 12 - 324x^{-3} = 12 - \frac{324}{x^3}$$

At min value of L, $\frac{dL}{dx} = 0$

$$\Rightarrow \quad 12 - \frac{324}{x^3} = 0$$

$$12x^3 - 324 = 0$$

$$12x^3 = 324$$

$$x^3 = 27$$

$$x = \sqrt[3]{27}$$

$$x = 3 \text{ cm}$$

Minimum $L = 12 \times 3 + \frac{162}{3^2} = 36 + 18$

$= 54 \text{ cm}$

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c) $$\frac{dL}{dx} = 12 - 324x^{-3}$$

$$\frac{d^2L}{dx^2} = +972x^{-4} = \frac{972}{x^4}$$

When $x = 3$, $\frac{d^2L}{dx^2} = \frac{972}{81} = 12 > 0$

$\therefore$ a minimum
