

Product Rule

Let $y = uv$ ① where u, v are functions of x

Let x change by a small amount δx
 This will cause a small change in u say δu ,
 in v say δv and therefore in y say δy .

$$y + \delta y = (u + \delta u)(v + \delta v) \quad ②$$

$$\textcircled{2} - \textcircled{1} \quad \therefore \delta y = (u + \delta u)(v + \delta v) - uv$$

$$\delta y = \cancel{uv} + v\delta u + u\delta v + \delta u \delta v - \cancel{uv}$$

$$\delta y = u\delta v + v\delta u + \delta u \delta v$$

$$\therefore \frac{\delta y}{\delta x} = u \frac{\delta v}{\delta x} + v \frac{\delta u}{\delta x} + \frac{\delta u}{\delta x} \delta v$$

Letting
 $\delta x \rightarrow 0$

$$\frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx} + \cancel{u \frac{du}{dx} \times 0}$$

$$\frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$$

The differential of a product of two functions is the first times the differential of the second plus the second times the differential of the first.

Examples

1) $y = x^2 \sin x$

$$\frac{dy}{dx} = x^2 \cos x + 2x \sin x$$

2) $y = x^3 e^x$

$$\begin{aligned}\frac{dy}{dx} &= x^3 e^x + 3x^2 e^x \\ &= e^x (x^3 + 3x^2)\end{aligned}$$

3) $y = (2x^3 + 3)^4 \sin x$

$$\begin{aligned}\frac{dy}{dx} &= (2x^3 + 3)^4 \cos x + 4(2x^3 + 3)^3 6x^2 \sin x \\ &= (2x^3 + 3)^4 \cos x + 24x^2 (2x^3 + 3)^3 \sin x \\ &= (2x^3 + 3)^3 [(2x^3 + 3) \cos x + 24x^2 \sin x]\end{aligned}$$

4) $y = (2x + 3)^5 x^2$

$$\begin{aligned}\frac{dy}{dx} &= 2x(2x + 3)^5 + x^2 \times 5(2x + 3)^4 (2) \\ &= (2x + 3)^4 [2x(2x + 3) + 10x^2]\end{aligned}$$

$$\begin{aligned}
 &= 2x(2x+3)^4[2x+3+5x] \\
 &= 2x(2x+3)^4(7x+3)
 \end{aligned}$$

Quotient Rule

$$\text{Let } y = \frac{u}{v} \quad \textcircled{1}$$

Then for a small change in x say δx

$$y + \delta y = \frac{u + \delta u}{v + \delta v} \quad \textcircled{2}$$

$\textcircled{2} - \textcircled{1}$

$$\delta y = \frac{u + \delta u}{v + \delta v} - \frac{u}{v}$$

$$\delta y = \frac{v(u + \delta u) - u(v + \delta v)}{(v + \delta v)v}$$

$$\delta y = \frac{\cancel{vu} + v\delta u - \cancel{uv} - u\delta v}{(v + \delta v)v}$$

$$\delta y = \frac{v\delta u - u\delta v}{(v + \delta v)v}$$

$\therefore \delta x$

$$\frac{\delta y}{\delta x} = \frac{v \frac{\delta u}{\delta x} - u \frac{\delta v}{\delta x}}{(v + \delta v)v}$$

$$\text{Letting } \delta x \rightarrow 0 \quad \frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{(v+0)v}$$

$$\frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$$

The differential of quotient is given by
 the bottom times the differential of the top
 minus the top times the differential of
 the bottom all over the bottom squared

Examples

1) $y = \frac{\sin x}{x^2}$

$$\frac{dy}{dx} = \frac{x^2 \cos x - 2x \sin x}{x^4}$$

$$= \frac{x(x \cos x - 2 \sin x)}{x^4}$$

$$= \frac{x \cos x - 2 \sin x}{x^3}$$

$$2) \quad y = \tan x = \frac{\sin x}{\cos x}$$

$$\frac{dy}{dx} = \frac{\cos x \cdot \cos x - \sin x(-\sin x)}{\cos^2 x}$$

$$\frac{dy}{dx} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

$$\left[\sec x = \frac{1}{\cos x} \quad \cosec x = \frac{1}{\sin x} \quad \cot x = \frac{1}{\tan x} \right]$$

$$\text{so } \frac{d}{dx} \tan x = \left(\frac{1}{\cos x} \right)^2 = \sec^2 x$$

Homework

Exercise 9D Q1, Q2

Exercise 9E Q1, Q2
