Leave blank

4. Prove by induction that, for $n \in \mathbb{Z}^+$,

$$\sum_{r=1}^{n} \frac{1}{r(r+1)} = \frac{n}{n+1}$$

(5)

$n = 1$ $\quad \frac{1}{1(1+1)} = \frac{1}{2} \qquad \frac{1}{1+1} = \frac{1}{2}$

$\therefore$ true when $n = 1$

Assume true for $n = k$

then $\sum_{r=1}^{k} \frac{1}{r(r+1)} = \frac{k}{k+1}$

$\frac{k+1}{k+2}$

Consider $n = k+1$

$$\sum_{r=1}^{k+1} \frac{1}{r(r+1)} = \frac{k}{k+1} + \frac{1}{(k+1)(k+2)}$$

$$= \frac{k(k+2) + 1}{(k+1)(k+2)}$$

$$= \frac{k^2 + 2k + 1}{(k+1)(k+2)}$$

$$= \frac{(k+1)^2}{(k+1)(k+2)}$$

$$= \frac{k+1}{k+2}$$

$$= \frac{(k+1)}{(k+1)+1}$$

This is the same formula with k replaced by $k+1$

$\therefore$ if true for $n=k$, also true for $n = k+1$

Since true for $n=1$, by mathematical induction it is true for all $n \in \mathbb{Z}^+$

N 3 4 6 9 4 A 0 8 2 8

Leave blank

8. Prove by induction that, for $n \in \mathbb{Z}^+$,

(a) $f(n) = 5^n + 8n + 3$ is divisible by 4,

(7)

(b) $\begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix}^n = \begin{pmatrix} 2n+1 & -2n \\ 2n & 1-2n \end{pmatrix}$

(7)

a) $f(n) = 5^n + 8n + 3$

when $n=1$ $f(1) = 5^1 + 8(1) + 3 = 16 = 4 \times 4$

$\therefore$ $f(n)$ divisible by 4 when $n=1$

Assume $f(n)$ divisible by 4 when $n=k$

$\therefore$ $f(k) = 5^k + 8k + 3 = 4h$

where h is an integer.

Consider $f(k+1) = 5^{k+1} + 8(k+1) + 3$

$= 5(5^k) + 8k + 8 + 3$

$= 5^k + 8k + 3 + 4(5^k) + 8$

$= 4h + 4(5^k) + 8$

$= 4(h + 5^k + 2)$

$\therefore$ divisible by 4

So if $f(k)$ is divisible by 4, so is $f(k+1)$

Since $f(1)$ is divisible by 4, by mathematical induction $f(n)$ is divisible by 4 for $n \in \mathbb{Z}^+$

(b) $\begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix}^n = \begin{pmatrix} 2n+1 & -2n \\ 2n & 1-2n \end{pmatrix}$

$\begin{pmatrix} 2(1)+1 & -2(1) \\ 2(1) & 1-2(1) \end{pmatrix}$

$= \begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix}$

$n=1 \quad \begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix}^1 = \begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix}$

$\therefore$ true for $n=1$

Assume true for $n=k$ $\quad \begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix}^k = \begin{pmatrix} 2k+1 & -2k \\ 2k & 1-2k \end{pmatrix}$

Consider $\begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix}^{k+1} = \begin{pmatrix} 2k+1 & -2k \\ 2k & 1-2k \end{pmatrix}\begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix}$

$$= \begin{pmatrix} 6k+3-4k & -4k-2+2k \\ 6k+2-4k & -4k-1+2k \end{pmatrix}$$

$$= \begin{pmatrix} 2k+3 & -2k-2 \\ 2k+2 & -2k-1 \end{pmatrix}$$

$$= \begin{pmatrix} 2(k+1)+1 & -2(k+1) \\ 2(k+1) & 1-2(k+1) \end{pmatrix}$$

Same matrix with k replaced by $k+1$

$\therefore$ if true for $n=k$ also true for $n=k+1$

Since true for $n=1$, by mathematical induction it is true for all $n \in \mathbb{Z}^+$

Leave blank

8. (a) Prove by induction that, for any positive integer n,

$$\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$$

(5)

(b) Using the formulae for $\sum_{r=1}^{n} r$ and $\sum_{r=1}^{n} r^3$, show that

$$\sum_{r=1}^{n}(r^3+3r+2) = \frac{1}{4}n(n+2)(n^2+7)$$

(5)

(c) Hence evaluate $\sum_{r=15}^{25}(r^3+3r+2)$

(2)

Leave blank

9. (a) Prove by induction that

$$\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$$

(6)

Using the standard results for $\sum_{r=1}^{n} r$ and $\sum_{r=1}^{n} r^2$,

(b) show that

$$\sum_{r=1}^{n} (r+2)(r+3) = \frac{1}{3}n(n^2 + an + b),$$

where a and b are integers to be found.

(5)

(c) Hence show that

$$\sum_{r=n+1}^{2n} (r+2)(r+3) = \frac{1}{3}n(7n^2 + 27n + 26)$$

(3)

Leave blank

9. A sequence of numbers $u_1, u_2, u_3, u_4, \ldots$ is defined by

$$u_{n+1} = 4u_n + 2, \quad u_1 = 2$$

Prove by induction that, for $n \in \mathbb{Z}^+$,

$$u_n = \frac{2}{3}\left(4^n - 1\right)$$

(5)

Leave blank

9. Prove by induction, that for $n \in \mathbb{Z}^+$,

(a) $\begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}^n = \begin{pmatrix} 3^n & 0 \\ 3(3^n - 1) & 1 \end{pmatrix}$,

(6)

(b) $\mathrm{f}(n) = 7^{2n-1} + 5$ is divisible by 12.

(6)

Leave blank

6. (a) Prove by induction

$$\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$$

(5)

(b) Using the result in part (a), show that

$$\sum_{r=1}^{n} (r^3 - 2) = \frac{1}{4}n(n^3 + 2n^2 + n - 8)$$

(3)

(c) Calculate the exact value of $\sum_{r=20}^{50} (r^3 - 2)$.

(3)

Leave blank

10. Prove by induction that, for $n \in \mathbb{Z}^+$,

$$f(n) = 2^{2n-1} + 3^{2n-1} \text{ is divisible by } 5.$$

(6)