

Implicit Differentiation

Example 1

$$x^2 + y^2 = 25$$

Find gradient when $x = 3$

$$y^2 = 25 - x^2$$

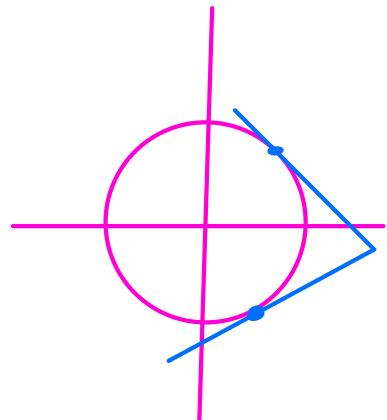
$$y = \pm \sqrt{25 - x^2}$$

$$y = \pm (25 - x^2)^{\frac{1}{2}}$$

$$\frac{dy}{dx} = \pm \frac{1}{2}(25 - x^2)^{-\frac{1}{2}}(-2x)$$

$$\frac{dy}{dx} = \pm \frac{x}{\sqrt{25 - x^2}}$$

$$\text{when } x = 3 \quad \frac{dy}{dx} = \pm \frac{3}{4}$$



Now using implicit differentiation

$$x^2 + y^2 = 25$$

$$\left[\frac{d}{dx} = \frac{d}{dy} \cdot \frac{dy}{dx} \right]$$

$$2x + 2y \frac{dy}{dx} = 0$$

diff
wrt
respect to x

$$2y \frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = -\frac{2x}{2y} = -\frac{x}{y}$$

$$\text{When } x = 3, y = 4 \quad \frac{dy}{dx} = -\frac{3}{4}$$

$$\text{When } x = 3, y = -4 \quad \frac{dy}{dx} = -\frac{3}{-4} = \frac{3}{4}$$

$$\text{Ex2} \quad y^3 + xy = 2$$

$$3y^2 \frac{dy}{dx} + x \frac{dy}{dx} + y = 0$$

$$\frac{dy}{dx} (3y^2 + x) = -y$$

$$\frac{dy}{dx} = -\frac{y}{3y^2 + x}$$

Find gradient at $(1, 1)$

$$\frac{dy}{dx} = -\frac{1}{3(1)^2 + 1} = -\frac{1}{4}$$

$$\text{Ex3} \quad x^3 + y^3 = 3xy$$

$$3x^2 + 3y^2 \frac{dy}{dx} = 3x \frac{dy}{dx} + 3y$$

$$3y^2 \frac{dy}{dx} - 3x \frac{dy}{dx} = 3y - 3x^2$$

$$\frac{dy}{dx} (3y^2 - 3x) = 3y - 3x^2$$

$$\frac{dy}{dx} = \frac{3y - 3x^2}{3y^2 - 3x} = \frac{y - x^2}{y^2 - x}$$

st pts when $\frac{dy}{dx} = 0 \Rightarrow y - x^2 = 0$
or $y = x^2$

But st pt must also be on curve so solve

$$\begin{cases} y = x^2 \\ x^3 + y^3 = 3xy \end{cases} \quad \text{simultaneously}$$

$$x^3 + x^6 = 3x^2 x^2$$

$$x^3 + x^6 = 3x^3$$

$$x^6 = 2x^3$$

$$x^6 - 2x^3 = 0$$

$$x^3(x^3 - 2) = 0$$

$$\Rightarrow x = 0 \text{ or } x = \sqrt[3]{2}$$

when $x = 0, y = 0$

when $x = \sqrt[3]{2}, y = 2^{2/3}$

st pts at $(0,0)$ and $(2^{\frac{1}{3}}, 2^{2/3})$

Ex 4 $\sin x + \sin y = 1$ $0 \leq x \leq \pi$
 $0 \leq y \leq \pi$

$$\cos x + \cos y \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{\cos x}{\cos y}$$

st pts when $\frac{dy}{dx} = 0 \Rightarrow \cos x = 0$
 $x = \frac{\pi}{2}$

Solve $\sin \frac{\pi}{2} + \sin y = 1$

$$1 + \sin y = 1$$

$$\sin y = 0$$

$$y = \sin^{-1}(0)$$

$$y = 0 \text{ or } \pi$$

st pts $(\frac{\pi}{2}, 0)$ and $(\frac{\pi}{2}, \pi)$

Exercise 4F $\frac{d}{dx} y^4 = 4y^3 \frac{dy}{dx}$

EXERCISE 4F

- 1** Differentiate each of the following with respect to x .
- (i) y^4
 - (ii) $x^2 + y^3 - 5$
 - (iii) $xy + x + y$
 - (iv) $\cos y$
 - (v) $e^{(y+2)}$
 - (vi) xy^3
 - (vii) $2x^2y^5$
 - (viii) $x + \ln y - 3$
 - (ix) $xe^y - \cos y$
 - (x) $x^2 \ln y$
 - (xi) $xe^{\sin y}$
 - (xii) $x \tan y - y \tan x$
- 2** Find the gradient of the curve $xy^3 = 5 \ln y$ at the point $(0, 1)$.
- 3** Find the gradient of the curve $e^{\sin x} + e^{\cos y} = e + 1$ at the point $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$.
- 4** (i) Find the gradient of the curve $x^2 + 3xy + y^2 = x + 3y$ at the point $(2, 0)$.
(ii) Hence find the equation of the tangent to the curve at this point.
- 5** Find the co-ordinates of all the stationary points on the curve $x^2 + y^2 + xy = 3$.
- 6** A curve has the equation $(x-6)(y+4) = 2$.
- (i) Find an expression for $\frac{dy}{dx}$ in terms of x and y .
 - (ii) Find the equation of the normal to the curve at the point $(7, -2)$.
 - (iii) Find the co-ordinates of the point where the normal meets the curve again.
 - (iv) By rewriting the equation in the form $y - a = \frac{b}{x-c}$ identify any asymptotes and sketch the curve.
- 7** A curve has the equation $y = x^x$ for $x > 0$.
- (i) Take logarithms to base e of both sides of the equation.
 - (ii) Differentiate the resulting equation with respect to x .
 - (iii) Find the co-ordinates of the stationary point, giving your answer to 3 decimal places.
 - (iv) Sketch the curve for $x > 0$.
- 8** (i) Show that the graph of $xy + 48 = x^2 + y^2$ has stationary points at $(4, 8)$ and $(-4, -8)$.
- (ii) By differentiating with respect to x a second time determine the nature of these stationary points.

$$1 \text{ i) } \frac{d}{dx} y^4 = 4y^3 \frac{dy}{dx}$$

$$1 \text{ ii) } \frac{d}{dx} (x^2 + y^3 - 5)$$

$$= 2x + 3y^2 \frac{dy}{dx}$$

$$\begin{aligned} 1 \text{ iii) } \frac{d}{dx} (xy + x + y) \\ = x \frac{dy}{dx} + y + 1 + \frac{dy}{dx} \end{aligned}$$

$$\begin{aligned} 1 \text{ iv) } \frac{d}{dx} (\cos y) \\ = -\sin y \frac{dy}{dx} \end{aligned}$$

Exercise 4F (Page 101)

1 (i) $4y^3 \frac{dy}{dx}$

(ii) $2x + 3y^2 \frac{dy}{dx}$

(iii) $x \frac{dy}{dx} + y + 1 + \frac{dy}{dx}$

(iv) $-\sin y \frac{dy}{dx}$

(v) $e^{(y+2)} \frac{dy}{dx}$

(vi) $y^3 + 3xy^2 \frac{dy}{dx}$

(vii) $4xy^5 + 10x^2y^4 \frac{dy}{dx}$

(viii) $1 + \frac{1}{y} \frac{dy}{dx}$

(ix) $xe^y \frac{dy}{dx} + e^y + \sin y \frac{dy}{dx}$

(x) $\frac{x^2}{y} \frac{dy}{dx} + 2x \ln y$

(xi) $e^{\sin y} + x \cos y e^{\sin y} \frac{dy}{dx}$

(xii) $\tan y + \frac{x}{\cos^2 y} \frac{dy}{dx} - (\tan x) \frac{dy}{dx} - \frac{y}{\cos^2 x}$

2 $\frac{1}{5}$

3 0

4 (i) -1

(ii) $x + y - 2 = 0$

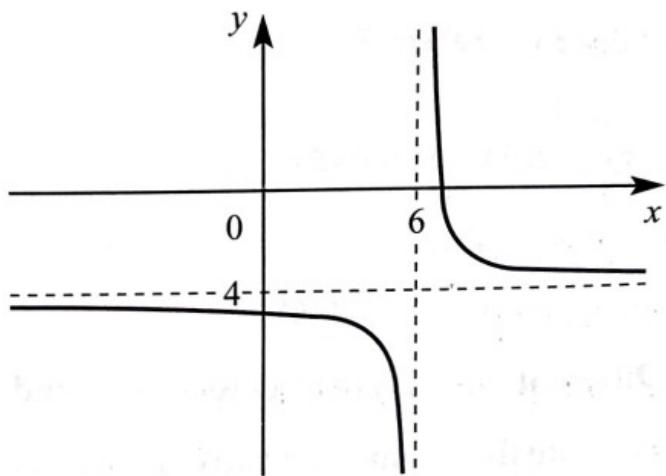
5 (1, -2) and (-1, 2)

6 (i) $\frac{y+4}{6-x}$

(ii) $x - 2y - 11 = 0$

(iii) $(2, -4\frac{1}{2})$

(iv)



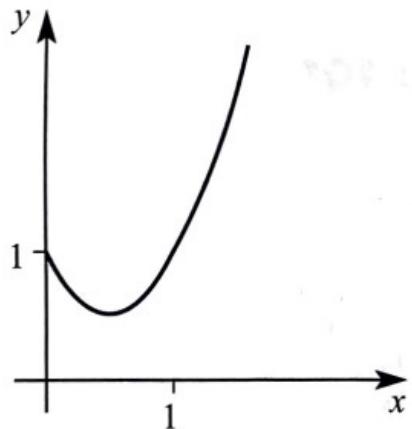
Asymptotes $x = 6, y = -4$

7 (i) $\ln y = x \ln x$

(ii) $\frac{1}{y} \frac{dy}{dx} = 1 + \ln x$

(iii) (0.368, 0.692)

(iv)



8 (ii) Max (4, 8), min (-4, -8)